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A PROPOSAL FOR THE C06 CHAPTER OF THE NAG ALGOL 68 LIBRARY

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A proposal for the C06 chapter of the NAG ALGOL 68 Library.

by

C.G. van der Laan*)

ABSTRACT

A proposal and future plans for the C06 chapter of the NAG ALGOL 68 library are provided. Operators in ALGOL 68 and their documentation are given for the problems connected with the Discrete Fourier Transform and the evaluation of trigonometric sums.

KEY WORDS & PHRASES: ALGOL 68 *implementation, Discrete Fourier Transform, Fast Fourier Transform, exponential sum, discrete harmonic analysis, trigonometric interpolation, discrete harmonic synthesis, trigonometric sum; Euler summation, Chebyshev sum, sum of orthogonal polynomials, Toeplitz matrix operations*

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CONTENTS

PREFACE

This publication is intended to serve the goals:

. to make the proposed included software free,
. to open a discussion over the proposed and future contents of the C06 chapter,
. to stimulate contributors.

The author is custodian for the C06 chapter of the NAG ALGOL 68 library.

Contributors to this work are my colleagues Mr. R.J. van Oosten and Mr. J.P. Hollenberg. The former programmed and documented a preliminary version of this collection of operators as well as the technical routine C06FFT; the latter worked out the final version and its test programs. The theoretical foundation of the included operators and a discussion of the available implementations in a variety of libraries was published by the author (in Dutch) in the MC Syllabus 29.1b: Colloquium Numerieke programmatuur.

Although a lot of good and fast FFT-routines are available in FORTRAN, we have programmed our own for portability reasons; users of the provided operators may interface to a FORTRAN version of C06FFT. A machine readable copy of the source texts can be obtained by sending a magnetic tape to the author.

Because the NAG ALGOL 68 library has absorbed the matrix/vector package TORRIX of VAN DER MEULEN & VELDHORST no explicit reference is given to this work.

Finally, I would like to thank the members of the working group Approximation of Functions, especially Dr. N.M. Temme, for the necessary environment and the Mathematical Centre for providing the opportunity to publish this proposal in their TW-reports series; Prof. L.M. Delves and Dr. G. Hodgson of NAG are kindly acknowledged for reading and commenting a previous version of the documentation units.

I INTRODUCTION TO THE C06 CHAPTER OF THE NAG ALGOL 68 LIBRARY

CONTENTS OF THIS INTRODUCTION

1. Scope of the Chapter

Provided are implementations related to

$$(1.1) \qquad s(t) = \sum_{k=l}^{u} f_k(t)$$

with t a parameter and l,u either finite or infinite.

2. Background of the Problems

The summation problems are distinguished according to the structure of the terms $f_k(t)$.

2.1 Types of summation problems

As a particular case of (1.1) we have

$$(2.1) \qquad f_k(t) = a_k \, \phi_k(t)$$

with $\underline{a}$ the coefficient vector of the series expansion of s with respect to the basis functions $\{\phi_k\}$.
A variety of implementations have emerged according to the available information and desired results.
Examples of $\{\phi_k\}$ are:

- polynomials (see chapter E02 of the NAG library);
- trigonometric functions (this chapter);
- B-splines (see chapter E02 of the NAG library).

With the vector $\underline{a}$ we mean $(\ldots a_{-n}, a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}, a_n, \ldots)$ or a finite part of it.

2.1.1 Summation of trigonometric functions

Within this context relation (1.1) reads

$$(2.2) \qquad s(t) = \sum_{k=l}^{u} \sigma_k \, e^{ikt}$$

The implementations given consider finite special cases of (2.2):

- evaluation of s for one argument (dyadic operators);
- evaluation of s for equidistant arguments (monadic operators).

2.1.2 General summation

So far no implementations are provided; in the ALGOL 68 report the routine EULER is given.

2.2 Accuracy

An implementation is generally used on samples in a digital computer, with its intrinsic limitations, whereas we are thinking in abstractions. The difference herein is covered by the accuracy concept, which may be subdivided into the error effects:

- residual or truncation error, i.e. the discrepancy between the sample information and the theoretical information of our problem;

. propagated error, i.e. the effect of perturbations;
. generated or rounding error, i.e. the effect of finite precision arithmetic.

In order to grasp these errors we use the concepts of the condition of a problem (section 2.2.1) and the growth of an algorithm (section 2.2.2). In each documentation unit bounds for the propagated and generated errors are given in terms of the condition and growth. Apart from the residual error a first order bound for the total error is the sum of the indicated errors.

Example.

Given the table

Formula	Propagated error	Generated error
C06EXPSUM a	$\|\underline{\Delta a}\|_2$ / max $(1/\sqrt{n}, \|\underline{a}\|_2)$	g_a * *small real*

we obtain for the error bound of $\{\underline{c} \mid c_j = \sum_{k=0}^{n} a_k e^{2\pi ikj/n}, \quad j = 0,1,\ldots,n-1\}$

$$\frac{\| c - \textit{C06EXPSUM a} \|_2}{\max (1, \|c\|_2)} \leq \text{propagated error + generated error.}$$

2.2.1 Condition of a problem

Consider a problem T, which transforms data $\underline{a}$ into $\underline{b}$, then the first order effect of the perturbation $\underline{\Delta a}$ of $\underline{a}$ is governed by

$$\underline{\Delta b} = \sum_j \frac{\partial Ta}{\partial a_j} \Delta a_j,$$

a weighted sum of the perturbations in the data.

The amplification or condition of the problem can be defined by

$$C = \frac{\|\underline{\Delta b}\|}{\|\underline{b}\|} \Big/ \frac{\|\underline{\Delta a}\|}{\|\underline{a}\|}.$$

If we choose the 2-norm then we obtain for the Discrete Fourier Transform (DFT)

$$C = 1.$$

2.2.2 Growth of an algorithm

Generally a process T in finite precision arithmetic is thought of as a process in exact arithmetic with perturbated parameters, $\underline{\Delta a}$. A bound for these perturbations can be obtained by backward error analysis; the factors involved are the so-called growth and the machine precision, ε. In other words, consider a process P for the above problem T in finite precision arithmetic, then there exists a $\underline{\Delta a}$, such that

$$P(\underline{a}) = fl(T(\underline{a})) = T(\underline{a}+\underline{\Delta a});$$

with the growth function, g, defined by

$$\frac{\|\underline{\Delta a}\|}{\|\underline{a}\|} = g * \varepsilon.$$

2.3 Data representation

Definitions:

o. a vector is called a periodic n-vector if $a_k = a_{k+n}$, for all integer k;

r. a vector is called a real periodic n-vector if (o) and the elements are real;

r.s. a vector is called a symmetric real periodic n-vector if (r) and $a_k = a_{n-k}$, for all integer k;

r.a. a vector is called an anti-symmetric real periodic n-vector if (r) and $a_k = -a_{n-k}$, for all integer k;

c. a vector is called a complex periodic n-vector if (o) and the elements are complex;

c.h. a vector is called a Hermitian complex periodic n-vector if (c) and $a_k = \bar{a}_{n-k}$, for all integer k;

c.a. a vector is called an anti-Hermitian complex periodic n-vector if (c) and $a_k = -\bar{a}_{n-k}$, for all integer k.

In the operators related to the DFT we will consider the significant part of a periodic n-vector; in the same way we will talk about a

symmetric, an anti-symmetric, a Hermitian or an anti-Hermitian vector. We use the notation:

$\underline{a}$, for either a real or complex vector;

$\underline{r}$, for a real vector;

$\underline{c}$, for a complex vector.

The data representation of an element a_k of a periodic n-vector is the number *a[LWB a + k MOD n]*, with *LWB a* a parameter free for choice.

3. Fourier Analysis and Synthesis

For the definition of DFT and IDFT and related terminology we adhere to [6]. For simplicity we restrict ourselves to periodic functions on [0,T), and trigonometric functions as basis functions.

Within this context relation (1.1) reads

$$(3.1) \qquad s(t) = \sum_{k=-\infty}^{\infty} \sigma_k \, e^{2\pi ikt/T}, \qquad t \in [0,T)$$

with the so-called Fourier coefficients

$$\sigma_k = \frac{1}{T} \int_0^T s(t) \, e^{-2\pi ikt/T} \, dt, \; k = \ldots,-n,\ldots,0,\ldots,n,\ldots$$

[11;509].

From relation (3.1) we considered two summation problems:

. Fourier analysis: given s obtain $\{\sigma_k\}$;

. Fourier synthesis: given $\{\sigma_k\}$ obtain s.

A variety of implementations may result according to the actual information used and delivered.

3.1 Fourier analysis

Generally only equidistant samples $\{s(2\pi k/n \mid k=0,1,\ldots,n-1\}$ are available and the first $\{\sigma_k \mid k=0,1,\ldots,n-1\}$ are desired.

The σ_k can be approximated by using attenuation factors τ_k as follows

$$(3.2) \qquad \sigma'_k = \tau_k * \sum_{j=0}^{n-1} s(2\pi j/n) \, \bar{W}_n^{kj}, \qquad k = 0,1,\ldots,n-1$$

with

$$w_n^{kj} = \exp(2\pi ikj/n)$$

and where the bar denotes complex conjugation and the prime the approximated value. The second factor of (3.2) apart from a factor is known as the Discrete Fourier Transform (DFT), [10;573].
For τ_k we may use a variety of possibilities, e.g.

τ_k	used information
1	$s(2\pi k/n)$, $k = 0,1,\ldots,n-1$.
$(\frac{\sin \pi k/n}{\pi k/n})^2$	$s(2\pi k/n)$, $k = 0,1,\ldots,n-1$, and function between samples is linear; σ'_k are the Fourier coefficients of the linear interpolated samples.

For more information about the attenuation factors, see [4].

Remarks

. The DFT can be handled by the monadic operators *C06EXPSUM* etc.
. The values of $\sin(2\pi k/n)$, $k = 0,1,\ldots,n-1$, can be obtained in s[0:n-1] by the call of the technical routine *C06SINTWI* as follows

C06SINTWI (n,s).

3.2 Fourier synthesis

Generally the first $\{\sigma_k \mid k=0,1,\ldots,n-1\}$ and either one argument t or n equidistant arguments $\{t_k \mid t_k = kT/n, k=0,1,\ldots,n-1\}$ are available while s(t) or $\{s(t_k)\}$ are desired.
This reduces to the problems:
. the evaluation of a trigonometric sum

$$\sum_{k=0}^{n-1} \sigma_k \, e^{2\pi ikt/T}$$

. the IDFT (the Inverse Discrete Fourier Transform)

$$\sum_{k=0}^{n-1} \sigma_k W_n^{kj}, \qquad j = 0,1,\ldots,n-1.$$

3.3 Applications

A collection of papers on the application of the DFT within the context of digital signal processing is given in [9,10]; with respect to time series analysis a concise survey of the backgrounds and a discussion of the application of the DFT to the calculation of sample covariance and cross-covariance functions, to the estimation of variance spectra and cross-spectra and very briefly to the implementation of moving average digital filters is given in [3]. In [6] the DFT is discussed and the following computational problems in (mainly) complex analysis where it can be fruitfully applied, are suggested: calculation of Fourier coefficients using attenuation factors; solution of Symm's integral equation in conformal mapping; trigonometric interpolation; determination of conjugate periodic functions and their application to Theodorsen's integral equation for the conformal mapping of simply and of doubly connected regions; determination of Laurent's coefficients with applications to numerical differentiation; generating functions and the numerical inversion of Laplace transforms; determination of the density of the zeros of high degree polynomials; convolution and its application to time series analysis, to the multiplication of polynomials and of large integers, and to fast Poisson solvers; manipulation of power series.

As an example we consider the evaluation of

$$\Phi(\omega) = \int_{-\infty}^{\infty} s(t) e^{-2\pi i \omega t} dt.$$

For the samples $\{\phi(j/T) \mid j=0,\pm1,\pm2,\ldots.\}$

the above integral reduces to

$$\Phi(j/T) = \int_{0}^{T} s_p(t) e^{-2\pi i j t/T} dt. \tag{3.3}$$

with the periodic alias function

$$s_p(t) = \sum_{k=-\infty}^{\infty} s(t+kT)$$

and T a free parameter. The integrals (3.3) can be evaluated by (3.2).

Remarks

. $s_p(t)$ is generally approximated by $s(t)$, $t \in [-T/2,T/2]$ with T such that the integration beyond $[-T/2,T/2]$ is negligible.

. With bandlimited functions we know already that $\Phi(\omega) = 0$ for $|\omega| > \Omega$.

This results in

$$-\Omega < j/T < \Omega, \quad j=0,\pm1,\ldots,\pm(n-1)$$

with $T = n\Delta t$ and Δt the sampling distance.
From a given $\Delta t \leq 1/(2\Omega)$ we can choose either the size n of the DFT or T. The behaviour of s and the desired accuracy of the approximation of s_p determines the total number of samples.

3.4 Implementation survey

For the moment only operators related to Fourier analysis and Fourier synthesis are available. These are grouped in four problem sets:

. evaluation of IDFT (monadic *C06EXPSUM* etc.):

$$\sum_{k=0}^{n-1} a_k \, e^{2\pi ikj/n}, \qquad j = 0,1,\ldots,n-1;$$

. evaluation of exponential sums (dyadic *C06EXPSUM* etc.):

$$f(\theta,\underline{a}) = \sum_{k=0}^{n} a_k \, z^k, \qquad z = e^{i\theta};$$

. evaluation of Discrete Harmonic Analysis (DHA) and Discrete Harmonic Synthesis (DHS) (monadic *C06TRGSUM* etc.):

from

$$f_j = \sum_{k=0}^{m}{}'' \; a_j \cos \pi kj/m + b_j \sin \pi kj/m, \quad j=0,1,\ldots,n-1 \text{ and } n=2*m$$

then the problems are characterized by

DHS: obtain $\underline{f}$ from $\underline{a}$ and $\underline{b}$

DHA: obtain $\underline{a}$ and $\underline{b}$ from f;

. evaluation of trigonometric sums (dyadic *C06TRGSUM* etc.):

$$f(\theta,\underline{a},\underline{b}) = \sum_{k=0}^{m} a_k \cos k\theta + b_k \sin k\theta.$$

Each problem set gives rise to a number of operators according to the data type and symmetries in the data: each problem set, and the details of the involved operators, is separately described in a documentation unit.

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II. DOCUMENTATION AND SOURCE TEXTS OF THE OPERATORS

II.1 *C06EXPSUM, C06EXPSUMHRM, C06EXPSUMANH.*

1. Purpose

The monadic operators

C06EXPSUM, C06EXPSUMHRM, C06EXPSUMANH

evaluate the Inverse Discrete Fourier Transform (IDFT) of a real or complex vector. Advantage has been taken of symmetry in the data in

C06EXPSUMHRM - data vector is Hermitian symmetric -

C06EXPSUMANH - data vector is skew-Hermitian symmetric.

IMPORTANT:

2. Specification (Algol 68)

```
MODE SCAL = REAL, COSCAL = COMPL;
MODE VEC        = REF[ ] SCAL,
   COVEC        = REF[ ] COSCAL;
OP C06EXPSUM    = (VEC r)     COVEC:
OP C06EXPSUM    = (COVEC c)   COVEC:
OP C06EXPSUMHRM = (VEC r)     VEC  :
OP C06EXPSUMHRM = (COVEC c)   VEC  :
OP C06EXPSUMANH = (VEC r)     VEC  :
OP C06EXPSUMANH = (COVEC c)   VEC  :
```

3. Description

The operators calculate

$$\sum_{k=0}^{n-1} a_k \, e^{2\pi ikj/n}, \qquad j=0,1\ldots,n-1$$

with a a real or complex vector. Used is the Cooley-Sande-Stockham algorithm; the auxiliary twiddle-factors are calculated by an extension of the Hopgood-Litherland algorithm.

4. References

See chapter introduction [1,2,7]

5. Parameters

General:

. n is the size of an array; for n is even we use m to denote n/2.

. The lower bound of the result equals the lower bound of the operand.

. Only the size of the operands matters: the k-th element of a vector a is assumed to be *a[LWB a + k]*, so the lower bound of the data representation of the vector does not matter and is free for choice.

. The operands are not preserved.

Formula	Operand	Result
C06EXPSUM a	a real or complex array variable with n elements.	a complex array variable with n elements.
C06EXPSUMHRM r	a real array variable with m+1 elements: the first elements of a symmetric n-vector, with n=2*m.	a real array variable with m+1 elements: the first elements of the resulting symmetric n-vector.
C06EXPSUMHRM c	a complex array variable with m+1 elements: the first elements of a Hermitian n-vector, with n=2*m.	a real array variable with n elements.
C06EXPSUMANH r	a real array variable with m+1 elements: the first elements of an anti-symmetric n-vector (the first and last element must contain zero), with n=2*m.	a real array variable with m+1 elements: the first elements of the imaginary part of the resulting complex Hermitian n-vector with zero real part. (The first and last element contain zero).
C06EXPSUMANH c	a complex array variable with m+1 elements: the first elements of a skew-Hermitian n-vector (the real parts of the first and last element must contain zero), with n=2*m.	a real array variable with n elements: the imaginary part of the resulting complex vector with zero real part.

6. Error indicators

In the event of an error condition being detected, the error routine: *c06fail* of mode *REF NAGFAIL*, is called with the parameters listed below. These are printed and in case the value of *c06fail* is *nagsoft* the

execution is continued (see in Introduction of the NAG manual the document on the ALGOL 68 error mechanism). The operators were designed with *nagsoft* as the user-friendly error-handling mechanism in mind.

parameter	message
1	OPERAND OF <operator name> OF WRONG SIZE The given array is of too small size, so the IDFT is an empty sum; the result yielded is the operand.
2	OPERAND OF <operator name> NOT <symmetry kind> The given part of the symmetric array is not of the symmetry kind expected by the operator, because of the symmetry and periodicity; the calculation is performed with an operand adapted to the operator, by setting elements to zero where appropriate.

7. Auxiliary routines

The used NAG library operators - all with a complex operand - are given in the following table.

Formula	Used NAG library operators	Used Torrix operators/ generators
C06EXPSUM r	none	*gencoarray*1, *widen*, *conj*
C06EXPSUM c	none	none
C06EXPSUMHRM r	m is odd: *C06EXPSUM*	*genarray*1, *gencoarray*1, *widen*, *conj*
	m is even: *C06EXPSUMHRM*	*genarray*1, *gencoarray*1, *widen*, *conj*
C06EXPSUMHRM c	none	*genarray*1, *gencoarray*1, *widen*, *conj*
C06EXPSUMANH r	m is odd: *C06EXPSUM*	*genarray*1, *gencoarray*1, *widen*, *conj*
	m is even: *C06EXPSUMHRM*	*genarray*1, *gencoarray*1, *widen*
C06EXPSUMANH c	*C06EXPSUMHRM*	*genarray*1, -, *<, *widen*

8. Timing

The time taken is proportional to p*n, where p is the sum of the prime factors of n.

9. Storage

The storage required by internally declared arrays is given in the following table.

Formula	Specification situation	Internally declared arrays
C06EXPSUM r	n is even	n+n÷2+1 complex elements.
	n is odd	n elements.
C06EXPSUM c		none.
C06EXPSUMHRM r	m is even	m÷2+1 real and complex elements.
	m is odd	m÷2+1 real and m complex elements.
C06EXPSUMHRM c		n real and m÷2+1 complex elements.
C06EXPSUMANH r	m is even	m÷2+1 real and complex elements.
	m is odd	m+1 real and m÷2+1 complex elements.
C06EXPSUMANH c		none.

10. Accuracy

Let us denote by:

- $\underline{a}$: the coefficient vector;
- a : the machine representation of the (measured) $\underline{a}$;
- $\underline{\Delta a}$: $\underline{a} - a$;
- g_a : the growth factor of order $\sim \sum_{i=1}^{j} p_i^{1.5}$

with $n = \prod_{i=1}^{j} p_i$ (given implicitly in [8]).

Error bounds (first order)

Formula	Propagated error	Generated error
C06EXPSUM a	$\|\underline{\Delta a}\|_2 / \max(1/\sqrt{n}, \|\underline{a}\|_2)$	g_a * *small real*

The bounds for *C06EXPSUMHRM*, and *C06EXPSUMANH* are similar.

11. Further comments

The related problems - the DFT apart from a factor 1/n -

$$\sum_{k=0}^{n-1} a_k \, e^{-2\pi i \, kj/n}$$

can be obtained as follows, where the parameters are prescribed as in 5. Parameters.

Operand	Formula
complex n-vector	*CONJ C06EXPSUM CONJ c*
Hermitian n-vector	*C06EXPSUMHRM CONJ* c
skew-Hermitian n-vector	- *C06EXPSUMANH CONJ* c
real n-vector	*CONJ C06EXPSUM* r
symmetric real n-vector	*C06EXPSUMHRM* r
anti-symmetric real n-vector	- *C06EXPSUMANH* r

12. Keywords

Discrete Fourier Transform.

Fast Fourier Transform.

Cooley-Sande-Stockham algorithm.

13. Examples

C06EXPSUM

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06EXPSUM'
   THIS PROGRAM CALCULATES APPROXIMATELY THE IDFT OF:
   - A COMPLEX VECTOR;
   - A REAL VECTOR.
   #
   'COVEC'C=GENCOVEC(2):=(1.0'I'2.0,3.0'I'4.0);
   WRITEF(($28A,2(L,16A,2(L,-D.DDQI-D.DD))$,
           "'C06EXPSUM' EXAMPLE PROGRAM.",
           "COMPLEX OPERAND:",
           C,
           "COMPLEX RESULT:",
           'C06EXPSUM'C));
```

```
   'VEC'R=GENVEC(4):=(1.0,0.0,3.0,4.0);
   WRITEF(($2L,13A,4(Q-D.DD),L,15A,4(L,-D.DDQI-D.DD)$,
           "REAL OPERAND:",
           R,
           "COMPLEX RESULT:",
           'C06EXPSUM'R))

'END'#OF 'C06EXPSUM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06EXPSUM' EXAMPLE PROGRAM.

COMPLEX OPERAND:
 1.00 I 2.00
 3.00 I 4.00
COMPLEX RESULT:
 4.00 I 6.00
-2.00 I-2.00

REAL OPERAND: 1.00 0.00 3.00 4.00
COMPLEX RESULT:
 8.00 I 0.00
-2.00 I-4.00
 0.00 I 0.00
-2.00 I 4.00
```

C06EXPSUMHRM

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06EXPSUMHRM' THIS PROGRAM
   CALCULATES APPROXIMATELY THE IDFT OF:
   - A COMPLEX HERMITIAN VECTOR;
   - A REAL SYMMETRIC VECTOR.
   #
   'COVEC'C=GENCOVEC(3):=(I.0'I'0.0,0.0'I'1.0,2.0'I'0.0);
   WRITEF(($31A,L,54A,3(L,-D.DDQI-D.DD),L,22A,4(Q-D.DDQ)$,
           "'C06EXPSUMHRM' EXAMPLE PROGRAM.",
           "SIGNIFICANT PART OF COMPLEX HERMITIAN VECTOR ON INPUT:",
           C,
           "REAL VECTOR DELIVERED:",
           'C06EXPSUMHRM'C));
```

```
    'VEC'R=GENVEC(3):=(1.0,2.0,3.0);
    WRITEF(($2L,51A,3(Q-D.DD),L,42A,3(Q-D.DD)$,
            "SIGNIFICANT PART OF REAL SYMMETRIC VECTOR ON INPUT:",
            R,
            "SIGNIFICANT PART OF REAL SYMMETRIC RESULT:",
            'C06EXPSUMHRM'R))
    'END'#OF 'C06EXPSUMHRM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06EXPSUMHRM' EXAMPLE PROGRAM.
SIGNIFICANT PART OF COMPLEX HERMITIAN VECTOR ON INPUT:
 1.00 I 0.00
 0.00 I 1.00
 2.00 I 0.00
REAL VECTOR DELIVERED: 3.00 -3.00 3.00 1.00
SIGNIFICANT PART OF REAL SYMMETRIC VECTOR ON INPUT: 1.00 2.00 3.00
SIGNIFICANT PART OF REAL SYMMETRIC RESULT: 8.00 -2.00 0.00
```

C06EXPSUMANH

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06EXPSUMANH' THIS PROGRAM
   CALCULATES APPROXIMATELY THE IDFT OF:
   - A COMPLEX ANTI-HERMITIAN VECTOR;
   - A REAL ANTI-SYMMETRIC VECTOR.
   #
   'COVEC'C=GENCOVEC(3):=(0.0'I'3.0,0.0'I'1.0,0.0'I'2.0);
   WRITEF(($31A,L,59A,3(L,-D.DDQI-D.DD),L,25A,4(Q-D.DD)$,
           "'C06EXPSUMANH' EXAMPLE PROGRAM.",
           "SIGNIFICANT PART OF COMPLEX ANTI-HERMITIAN VECTOR ON INPUT:",
           C,
           "DELIVERED IMAGINARY PART:",
           'C06EXPSUMANH'C));
'VEC'R=GENVEC(3):=(0.0,2.0,0.0);
WRITEF(($2L,56A,3(Q-D.DD),L,37A,3(Q-D.DD)$,
        "SIGNIFICANT PART OF REAL ANTI-SYMMETRIC VECTOR ON INPUT:",
        R,
        "SIGNIFICANT PART OF IMAGINARY RESULT:",
        'C06EXPSUMANH'R))
'END'#OF 'C06EXPSUMANH' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06EXPSUMANH' EXAMPLE PROGRAM.
SIGNIFICANT PART OF COMPLEX ANTI-HERMITIAN VECTOR ON INPUT:
 0.00 I 3.00
 0.00 I 1.00
 0.00 I 2.00
DELIVERED IMAGINARY PART: 7.00 1.00 3.00 1.00
SIGNIFICANT PART OF REAL ANTI-SYMMETRIC VECTOR ON INPUT: 0.00 2.00 0.00
SIGNIFICANT PART OF IMAGINARY RESULT: 0.00 4.00 0.00
```

14. Source texts

```
OP  C06EXPSUM =( COVEC c) COVEC :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×c[k+ LWB c], j=0,1, ... ,n-1
                   k=0
                   with n= SIZE c,w[j,k]=exp(0 I j×k×2×pi/n),
input:             see above formula. c is not preserved.
results:           the idft of c is delivered
                   with bounds similar to those of c.
exception handling:if  SIZE c<1 then c is delivered and c06fail is
                   called.#

IF  INT n= SIZE c;n>1
THEN c06fft(c[ AT 0]);c
ELSE c06fail(1,"expsumoperand of  C06EXPSUM  of wrong size");c

FI ,
```

```
OP  C06EXPSUM =( VEC r) COVEC :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×r[k+ LWB r], j=0,1, ... ,n-1
                   k=0

                   with  n= SIZE r, w[j,k]=exp(0 I 2×pi×j×k/n).
input:             see above formula. r is not preserved.
results:           the idft of (real) r with similar bounds as r.
exception handling:if  SIZE r<1 then r, widened to a  COVEC , is
                   delivered and c06fail is called.#

IF  INT n= SIZE r;n>0
THEN  INT l= LWB r,u= UPB r;
   IF  ODD n
   THEN  COVEC xy= WIDEN r[ AT 0];
      c06fft(xy);
      INT uu:=n-1;
      FOR ll WHILE ll<uu
      DO  REF  COSCAL xyl=xy[ll],xyu=xy[uu];
         COSCAL rs=(xyl+ CONJ xyu)/ WIDEN 2;
         (xyl:=rs,xyu:= CONJ rs);
         uu-:=1
      OD ;
      xy[ AT 1]
   ELSE  INT n2=n OVER 2,n4=n OVER 4, INT j:=-1;
      COVEC xy=gencoarrayl(0,n-1);
      FOR i FROM 1 BY 2 TO u-1
      DO xy[j+:=1]:=r[i] I r[i+1] OD ;
      c06fft(xy[0:n2-1 AT 0]);
      BEGIN  SCAL normdiv2= WIDEN 1/ WIDEN 2;

         COVEC wn=gencoarrayl(0,n4);
         c06initw(n,wn);
         FOR i TO n4
         DO  REF  COSCAL s=xy[i],t=xy[n2-i],
            COSCAL wni=wn[i];
            COSCAL p=(s+ CONJ t)×normdiv2,
            q=(s- CONJ t)×(im OF wni I -re OF wni)×normdiv2;
            (s:=p+q,t:= CONJ (p-q))
         OD ;
         ( SCAL s=re OF xy[0],t=im OF xy[0];
         (xy[0]:= WIDEN (s+t),xy[n2]:= WIDEN ( s-t)))
      END ;
      INT uu:=n-1;
      FOR ll WHILE ll<uu
      DO xy[uu]:= CONJ xy[ll];uu-:=1 OD ;
      xy[ AT 1]
   FI
ELSE c06fail(1,"expsumoperand of  C06EXPSUM  of wrong size");
   WIDEN r
FI ,
```

```
 OP  C06EXPSUMHRM =( COVEC c) VEC :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×c[ LWB c+k], j=0,1, ... ,n-1
                   k=0

                   with n=n2×2, the size of the complete hermitian
                   vector c, (i.e. c[ LWB c+k]= CONJ c[ LWB c+n-k],
                   k=1,2, ... , n-1) and w[j,k]=exp(0 I 2×pi×j×k/n).
input:             c[ LWB c: LWB c+n2] is assumed to be supplied.
                   c is not preserved. for odd n use  C06EXPSUM .
results:           the idft of c is a real vector and is delivered
                   as a  VEC  with the lower bound similar to that of
                   c and with  LWB c+n-1 as upper bound.
exception handling:if the first or last element of the given c have
                   nonzero imaginairy parts (so the complete vector
                   is not hermitian symmetric) then the calculation is
                   done with these parts put to zero and c06fail is
                   called. if n<2 then a  VEC  with bounds  LWB c and
                   LWB c+n-1 is delivered and c06fail is called.#

 IF  INT l= LWB c,u= UPB c; INT n2=u-1; INT n=n2×2;n2>1
 THEN  IF  SCAL zero= WIDEN 0;
       im OF c[1]=zero AND im OF c[u]=zero
    THEN  INT n4=n2 OVER 2;
       COVEC czer=c[1:1+n2-1 AT 0];
       COVEC wn=gencoarray1(0,n4);
       c06initw(n,wn);
       czer[0]:=( SCAL c0=re OF czer[0],cn2=re OF c[u];
       (c0+cn2) I (c0-cn2));
       FOR k TO n4
       DO  REF  COSCAL ck=czer[k],cn2k=czer[n2-k];
          COSCAL s= ck+ CONJ cn2k,
                  t=(ck- CONJ cn2k)×wn[k]×(0 I 1);
          (ck:=s+t,cn2k:= CONJ (s-t))
       OD ;
       c06fft(czer);
       VEC result=genarray1(0,n-1);
       INT j:=-1;
       FOR k BY 2 TO n-1
       DO result[k-1]:=re OF czer[j+:=1];
           result[k  ]:=im OF czer[j]
       OD ;
       result[ AT 1]
    ELSE c06fail(2,"exphrmoperand of  C06EXPSUMHRM  not hermitian");
       im OF c[1]:=im OF c[u]:=zero; C06EXPSUMHRM c
    FI
 ELSE c06fail(1,"exphrmoperand of  C06EXPSUMHRM  of wrong size");
    genarray1(1,1+n-1)
 FI ,
```

```
  OP  C06EXPSUMHRM =( VEC r) VEC :

 #purpose:          approximately calculated is

                    n-1
                    sum w[j,k]×r[ LWB r+k], j=0,1, ... ,n-1
                    k=0

                    with n=n2×2, the size of the complete symmetric
                    vector r, (i.e. r[ LWB r+k]=r[ LWB r+n-k],
                    k=1,2, ... ,n-1) and w[j,k]=exp(0 I 2×pi×j×k/n).
 input:             r[ LWB r: LWB r+n2] is assumed to be supplied.
                    r is not preserved. for odd n use  C06EXPSUM .
 results:           the dft of r is again real and symmetric, so only
                    the first n2+1 elements are delivered with bounds
                    similar to those of r.
exception handling:if n<2 then c06fail is called and the original
                    vector is delivered.#

 IF  INT n2= SIZE r-1;n2>0
 THEN  SCAL oddsum:= WIDEN 0, VEC rzer=r[ AT 0];
    FOR k BY 2 TO n2-1
    DO oddsum+:=2×rzer[k] OD ;
    IF  ODD n2
    THEN oddsum+:=rzer[n2];
       COVEC c=gencoarrayl(0,n2-1);
       INT j:=0;c[0]:= WIDEN rzer[0];
       FOR k FROM 2 BY 2 TO n2
       DO  COSCAL cs=c[j+:=1]:=rzer[k] I (rzer[k+1]-rzer[k-1]);
          c[n2-j]:= CONJ cs
       OD ;
       rzer[0:n2-1 AT 0]:=re OF ( C06EXPSUM c)
    ELSE  INT n4=n2 OVER 2; COVEC c=gencoarrayl(0,n4);
       INT j:=0;c[0]:= WIDEN rzer[0];
       FOR k FROM 2 BY 2 TO n2-2
       DO c[j+:=1]:=rzer[k] I (rzer[k+1]-rzer[k-1]) OD ;
       c[n4]:= WIDEN rzer[n2];
       rzer[0:n2-1 AT 0]:= C06EXPSUMHRM c
    FI ;
    SCAL evensum=rzer[0];
    (rzer[0]+:=oddsum,rzer[n2]:=evensum-oddsum);
    INT m=n2 OVER 2;
    VEC sintwi=genarrayl(0,m);
    c06sintwi(2×n2,sintwi);
    INT uu:=n2;
    SCAL two= WIDEN 2;
    FOR k TO m
    DO  REF  SCAL s=rzer[k],t=rzer[uu-:=1];
       SCAL sk=s+t,tk=(s-t)/(two×sintwi[k]);
       (s:=(sk+tk)/two,t:=(sk-tk)/two)
    OD ;
    r
ELSE c06fail(1,"exphrmoperand of  C06EXPSUMHRM  of wrong size");r
FI ,
```

```
 OP  CO6EXPSUMANH =( COVEC c) VEC :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×c[ LWB c+k], j=0,1, ... ,n-1
                   k=0

                   with n=n2×2, the size of the complete anti-
                   hermitian vector c,
                   (i.e. c[ LWB c+k]=- CONJ c[ LWB c+n-k],
                   k=1,2, ... ,n-1) and w[j,k]=exp(0 I 2×pi×j×k/n).
 input:            c[ LWB c: LWB c+n2] is assumed to be supplied.
                   c is not preserved. for odd n use  CO6EXPSUM .
 results:          the dft of c is a vector with a zero real part.
                   the imaginary part is delivered as a  VEC  with the
                   same lower bound as c and with  LWB c+n-1 as upper
                   bound.
exception handling:if the first or last element of the given c have
                   non zero real parts (so the complete vector is not
                   anti-hermitian symmetric) then the calculation is
                   done with these parts put to zero and c06fail is
                   called. if n<2 then a  VEC  with bounds  LWB c and
                   LWB c+n-1 is delivered and c06fail is called.#

 IF  INT l= LWB c,u= UPB c; INT n=(u-1)×2;n>1
 THEN  IF  SCAL zero= WIDEN 0;
       re OF c[l]=zero AND re OF c[u]=zero
    THEN  CO6EXPSUMHRM (-c×<(0 I 1))
    ELSE c06fail(2,
       "expanhoperand of  CO6EXPSUMANH  not anti-hermitian");
       re OF c[l]:=re OF c[u]:=zero;
       CO6EXPSUMHRM (-c×<(0 I 1))
    FI
 ELSE c06fail(1,"expanhoperand of  CO6EXPSUMANH  of wrong size");
    genarrayl(1,l+n-1)
 FI ,


 OP  CO6EXPSUMANH =( VEC r) VEC :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×r[ LWB r+k], j=0,1, ... ,n-1
                   k=0

                   with n=n2×2, the size of the complete anti-
                   symmetric vector r,
                   (i.e. r[ LWB r+k]=-r[ LWB r+n-k], k=1,2, ... , n-1)
                   and w[j,k]=exp(0 I 2×pi×j×k/n).
```

```
 input:              r[ LWB r: LWB r+n2] is assumed to be supplied.
                     r is not preserved. for odd n use  C06EXPSUM .
 results:            the dft of r is a complex hermitian vector with a
                     zero real part.
                     only the first n2+1 elements of the imaginary part
                     are delivered as a vec, with bounds similar to
                     those of r.
                     note that the first and the last element of the
                     delivered vec are zero.
exception handling:if the first and the last elements of the given r
                     are not zero then the calculation is done with
                     these parts put to zero and c06fail is called. if
                     n<2 then c06fail is called and the original vector
                     is delivered.#

 IF  INT l= LWB r,u= UPB r; INT n2=u-1;n2>0
 THEN  IF  SCAL zero= WIDEN 0;
       r[l]=zero AND r[u]=zero
    THEN  VEC rzer=r[ AT 0]; SCAL two= WIDEN 2;
       IF  ODD n2
       THEN  COVEC c=gencoarrayl(0,n2-1);
          INT j:=0;c[0]:= WIDEN (two×rzer[1]);
          FOR k FROM 2 BY 2 TO n2
          DO  COSCAL cs=c[j+:=1]:=(rzer[k+1]-rzer[k-1]) I rzer[k];
                c[n2-j]:= CONJ cs
          OD ;
          rzer[0:n2-1 AT 0]:=re OF ( C06EXPSUM c)
       ELSE  INT n4=n2 OVER 2; COVEC c=gencoarrayl(0,n4);
          INT j:=0;c[0]:= WIDEN (two×rzer[1]);
          FOR k FROM 2 BY 2 TO n2-2
          DO c[j+:=1]:=(rzer[k+1]-rzer[k-1]) I rzer[k] OD ;
          c[n4]:= WIDEN (-two×rzer[n2-1]);
          rzer[0:n2-1 AT 0]:= C06EXPSUMHRM c
       FI ;
       INT m=n2 OVER 2;
       VEC sintwi=genarrayl(0,m);
       c06sintwi(2×n2,sintwi);
       INT uu:=n2;
       FOR k TO m
       DO  REF  SCAL s=rzer[k],t=rzer[uu-:=1];
          SCAL sk=t-s,tk=(t+s)/(two×sintwi[k]);
          (s:=(sk+tk)/two,t:=(tk-sk)/two)
       OD ;
       rzer[0]:=rzer[n2]:=zero;
       r
       ELSE c06fail(2,
          "expanhoperand of  C06EXPSUMANH  not anti-symmetric");
       r[1]:=r[u]:=zero; C06EXPSUMANH r
    FI
ELSE c06fail(1,"expanhoperand in  C06EXPSUMANH  of wrong size");r
FI
```

II.2 *C06EXPSUM* (dyadic)

1. Purpose

The dyadic operators

C06EXPSUM

evaluate an exponential sum with real or complex coefficients.
IMPORTANT:....

2. Specification (Algol 68)

```
MODE SCAL = REAL, COSCAL = COMPL;
MODE VEC    = REF[ ]SCAL,
     COVEC  = REF[ ]COSCAL;
OP C06EXPSUM = (SCAL t, VEC r)   COSCAL:
OP C06EXPSUM = (SCAL t, COVEC c) COSCAL:
PRIO C06EXPSUM = 8.
```

3. Description

The operators calculate

$$f(\theta, \underline{a}) = \sum_{k=0}^{n} a_k z^k, \quad z=e^{i\theta},$$

with $\underline{a}$ a real or complex vector. The problem is reduced to the problem of evaluating trigonometric sums - i.e. sine and cosine sums - by considering the real and imaginary parts. Used is the Clenshaw algorithm with the modifications due to Reinsch.

4. References

[1] OLIVER, J.
An error analysis of the modified Clenshaw method for evaluating Chebyshev and Fourier series.
JIMA, vol. 20, 379-391. 1977.

5. Parameters

General:

. Both operands are preserved.
. The result is a complex constant: the exponential sum.
. Only the size of the right operand matters: the k-th element of a vector a is assumed to be represented by *a[LWB a + k]* so the lower bound of the data representation of the vector does not matter and is free for choice.

Left operand *t*: the angle θ, a real constant (it is advised to supply a value within $[-\pi,\pi)$).
Right operand : a real or complex array with n+1 elements.

6. Error indicators

In the event of an error condition being detected, the error routine: *c06fail* of mode *REF NAGFAIL*, is called with the parameters listed below. These are printed and in case the value of *c06fail* is *nagsoft* the execution is continued (see in Introduction of the NAG manual the document on the ALGOL 68 error mechanism). The operators were designed with *nagsoft* as the user-friendly error-handling mechanism in mind.

parameter	message
1	VECTOR OPERAND OF *C06EXPSUM* OF WRONG SIZE The size of the given array is smaller than zero; the result yielded is zero as value for the empty sum.

7. Auxiliary routines None.

8. Timing

The time taken is proportional to n.

9. Storage No auxiliary arrays are declared.

10. Accuracy

Let us denote by:

. $\underline{a}$: the coefficient vector;

. θ : the angle;

. c_a : the condition number $\left(\sum_{k=0}^{n} \max(1,|a_k|)\right)/\max(1,|f(\theta,\underline{a})|)$;

. c_θ : the condition number $|\max(1,|\theta|)\frac{\partial f}{\partial \theta}|/\max(1,|f(\theta,\underline{a})|)$;

. g_a : the growth factor (conjectured of order n);

. *a* : the machine representation of (the measured) $\underline{a}$;

. *t* : the machine representation of (the measured) θ;

. $\delta\underline{a}$: vector of componentswise errors:

$$\delta a_k = |a_k - a[LWB\ a + k]|/\max(1, |a_k|)$$

for all appropriate k;

. $\delta\theta$: $|t - \theta|/\max(1, |\theta|)$.

Error bounds (first order)

Formula	Propagated error	Generated error
t C06EXPSUM a	$c_a * \|\|\delta\underline{a}\|\|_\infty + c_\theta * \delta\theta$	$c_a * g *$ *small real*

11. Further comments None.

12. Keywords

Exponential sum.

Clenshaw-Reinsch algorithm.

13. Examples

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06EXPSUM' (DYADIC) THE PROGRAM
   CALCULATES APPROXIMATELY

    1
   SUM  A[K]*EXP(0'I'K*T)
   K=-1

   #
   'SCAL' ANGLE=PI/2;
   'VEC'APOS=(GENVEC(2):=(0.5,2.0))['AT'0],
        ANEG=(GENVEC(2):=(0.5,2.0))['AT'0];
   WRITEF(($28A,L,6A,Q-D.DD,L,34A,3(Q-D.DD),L,25A,Q-D.DDQI-D.DD$,
           "'C06EXPSUM' EXAMPLE PROGRAM.",
           "ANGLE:",ANGLE,
           "COEFFICIENTS A[-1],A[0] AND A[1]:",
           ANEG[1],ANEG[0]+APOS[0],APOS[1],
           "VALUE OF EXPONENTIAL SUM:",
           -ANGLE'C06EXPSUM'ANEG+ANGLE'C06EXPSUM'APOS));

   'COVEC'APLS=(GENCOVEC(2):=(0.5'I'0.25,0.5'I'-0.5))['AT'0],
          AMIN=(GENCOVEC(2):=(0.5'I'0.25,0.5'I'-0.5))['AT'0];
   WRITEF(($2L,6A,Q-D.DD,L,34A,3(L,-D.DDQI-D.DD),L,25A,Q-D.DDQI-D.DD$,
           "ANGLE:",ANGLE,
           "COEFFICIENTS A[-1], A[0] AND A[1]:",
           AMIN[1],AMIN[0]+APLS[0],APLS[1],
           "VALUE OF EXPONENTIAL SUM:",
           -ANGLE'C06EXPSUM'AMIN+ANGLE'C06EXPSUM'APLS))
'END'#OF 'C06EXPSUM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06EXPSUM' EXAMPLE PROGRAM.
ANGLE:  1.57
COEFFICIENTS A[-1], A[0] AND A[1]: 2.00 1.00 2.00
VALUE OF EXPONENTIAL SUM: 1.00 I 0.00

ANGLE:  1.57
COEFFICIENTS A[-1], A[0] AND A[1]:
 0.50 I-0.50
 1.00 I 0.50
 0.50 I-0.50
VALUE OF EXPONENTIAL SUM:  1.00 I 0.50
```

14. Source texts

```
OP   C06EXPSUM =( SCAL t, COVEC c) COSCAL :

#purpose:          approximately calculated is the exponential sum

                    n
                   sum c[ LWB c+k]×z↑k
                   k=0

                   with z=exp(0 I t) and n= SIZE c-1.
input:             left operand: angle t of  MODE   SCAL ; it is
                   advised to take t in the interval [-pi,pi).
                   right operand: coefficient vector c of  MODE
                   COVEC .
result:            the result of the above, possibly empty, sum of
                   MODE   COSCAL .
exception handling:if n<0 then c06fail is called and zero
                   is delivered.#

IF   SIZE c>0  THEN
   SCAL  sinserr,cosserr,sinseri,cosseri;
   (c06ser( re  OF c,cosserr,sinserr,t),
   c06ser( im  OF c,cosseri,sinseri,t));
   (cosserr-sinseri) I (sinserr+cosseri)
ELSE  c06fail(1,"expsumvector operand of c06expsum of wrong size");
   WIDEN  WIDEN 0
FI ,


OP   C06EXPSUM =( SCAL t, VEC r) COSCAL :

#purpose:          approximately calculated is the exponential sum

                    n
                   sum r[ LWB r+k]×z↑k
                   k=0

                   with z=exp(0 I t) and n= SIZE r-1.
input:             left operand: angle t of  MODE   SCAL ; it is
                   advised to take t in the interval [-pi,pi).
                   right operand: coefficient vector r of  MODE
                   VEC .
result:            the result of the above, possibly empty, sum of
                   MODE   COSCAL .
exception handling:if n<0 then c06fail is called and zero
                   is delivered.#

IF   SIZE r>0  THEN
   SCAL  sinser,cosser;
   c06ser(r,cosser,sinser,t);
   cosser I sinser
ELSE  c06fail(1,"expsumvector operand of c06expsum of wrong size");
   WIDEN  WIDEN 0
FI
```

II.3 *C06TRGSUM, C06COSSUM, C06SINSUM*

1. Purpose

The monadic operators

C06TRGSUM, C06COSSUM, C06SINSUM

evaluate the Discrete Harmonic Analysis (DHA) and Discrete Harmonic Synthesis (DHS).

Advantage has been taken of zeros in the data in

C06COSSUM - sine coefficients are zero (a Discrete Cosine Transform (DCT))

C06SINSUM - cosine coefficients are zero (a Discrete Sine Transform (DST)).

IMPORTANT:....

2. Specification (Algol 68)

```
MODE SCAL     = REAL, COSCAL = COMPL;
MODE VEC      = REF[ ] SCAL,
     COVEC    = REF[ ]COSCAL;
OP C06TRGSUM  = (VEC r)     COVEC:
OP C06TRGSUM  = (COVEC ab)  VEC  :
OP C06COSSUM  = (VEC a)     VEC  :
OP C06SINSUM  = (VEC b)     VEC  :
```

3. Description

Given the relation

$$f_j = \sum_{k=0}^{m}{}'' (a_j*\cos(\pi*k*j/m)+b_j*\sin(\pi*k*j/m)),\ j=0,1,\ldots,n-1,\ n=2*m,$$

then the problems are characterized by:

DHS: obtain f from a and b,

DHA: obtain a and b from f,

DCT: obtain f from a; b is zero,

DST: obtain f from b; a is zero.

Used is the Cooley-Sande-Stockham algorithm for the DFT because the above relation is equivalent to

$$a_j+ib_j = 1/m \sum_{k=0}^{n-1} f_k * e^{2\pi ikj/n}, \quad j=0,1,\ldots,m.$$

4. References

See chapter introduction: [1,2].

5. Parameters

General:

. n is the size of the array of real variables *f* of even size; we use m to denote n/2.

. The lower bound of the result equals the lower bound of the operand.

. Only the size of the operands matters: the k-th element of a vector c is assumed to be represented by *c[LWB c + k]*, so the lower bound of the data representation of the vector does not matter and is free for choice.

. The operands are not preserved.

. The first and last element of *b* must contain zero.

Formula	Operand	Result
C06TRGSUM f	a real array variable with n elements: f.	a complex array variable with m+1 elements: the real part contains a and the imaginary part contains b.
C06TRGSUM ab	a complex array variable with m+1 elements: the real part contains a and the imaginary part contains b.	a real array variable with n elements: f.
C06COSSUM a	a real array variable with m+1 elements: a.	a real array variable with m+1 elements: the DCT of the operand.
C06SINSUM b	a real array variable with m+1 elements: b.	a real array variable with m+1 elements: the DST of the operand.

6. Error indicators

In the event of an error condition being detected the error routine: *c06fail* of mode *REF NAGFAIL*, is called with the parameters listed below. These are printed and in case the value of *c06fail* is *nagsoft* the execution is continued (see in Introduction of the NAG manual the document on the ALGOL 68 error mechanism). The operators were designed with *nagsoft* as the user-friendly error-handling mechanism in mind.

parameter	message
1	OPERAND OF <operator name> OF WRONG SIZE The given array is of too small size; the result yielded is the operand.
2	OPERAND OF <operator name> CONTAINS NONZERO FIRST AND/OR LAST SINE COEFFICIENT The calculation is performed with the nonzero elements in question overwritten with zeros.
3	OPERAND (REAL VECTOR) OF *C06TRGSUM* IS OF ODD LENGTH The calculation is performed with an adapted operand; the smaller of the first and the last element is discarded.

7. Auxiliary routines

The used NAG library operators are given in the following table.

Formula	Used NAG library operators	Used Torrix operators/generators
C06TRGSUM f	*C06EXPSUM*	*gencoarrayl*,/<, *gencoarray* 1
C06TRGSUM ab	*C06EXPSUMHRM*	*genarrayl*,/<, *conj*, *genarray* 1, *widen*
C06COSSUM a	*C06EXPSUMHRM*	/<
C06SINSUM b	*C06EXPSUMANH r*	/<

8. Timing

The time taken is proportional to s*m, where s is the sum of the factors of m.

9. Storage No auxiliary arrays are declared.

10. Accuracy

The accuracy is determined by the DFT because of the second relation in the description.

Let us denote by

. c : the machine representation of a (measured) vector $\underline{c}$;

. $\underline{\Delta c}$: $|c - \underline{c}|$.

Error bounds (first order).

Formula	Propagated error	Generated error
C06TRGSUM f	$\|\underline{\Delta f}\|_2 / \max(1/\sqrt{n}, \|\underline{f}\|_2)$	g_f* *small real*
C06TRGSUM ab	$\|\underline{\Delta a}\|_2 / \max(1/\sqrt{n}, \|\underline{a}\|_2) + \|\underline{\Delta b}\|_2 / \max(1/\sqrt{n}, \|\underline{b}\|_2)$	(g_a+g_b)* *small real*
C06COSSUM a	$\|\underline{\Delta a}\|_2 / \max(1/\sqrt{n}, \|\underline{a}\|_2)$	g_a* *small real*
C06SINSUM b	$\|\underline{\Delta b}\|_2 / \max(1/\sqrt{n}, \|\underline{b}\|_2)$	g_b* *small real*

The growth factors g_f, g_a, g_b are of the order of magnitude

$$\sim \sum_i p_i^{1.5}$$

with p_i the factors of m.

11. Further comments None.

12. Keywords

Discrete harmonic analysis.
Trigonometric interpolation.
Discrete harmonic synthesis.
Evaluation of trigonometric sums.
Fast Fourier Transform.
Cooley-Sande-Stockham algorithm.

13. Examples

C06TRGSUM

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06TRGSUM' THIS PROGRAM CALCULATES
   APPROXIMATELY THE DISCRETE HARMONIC ANALYSIS AND THE DISCRETE
   HARMONIC SYNTHESIS.
   #
   'VEC'F=GENVEC(4):=(1.0,0.0,3.0,4.0);
   WRITEF(($28A,L,28A,4(Q-D.DD),L,8A,L,42A,3(L,-D.DD17QIQ-D.DD)$,
           "'C06TRGSUM' EXAMPLE PROGRAM.",
           "DISCRETE FUNCTION, ON INPUT:",
           F,
           "RESULTS:",
           "COSINE COEFFICIENTS:    SINE COEFFICIENTS:",
           'C06TRGSUM'F));
   'COVEC'AB=GENCOVEC(5):=(2.0'I'0.0,0.0'I'1.0,0.0'I'2.0,2.0'I'3.0,
                           4.0'I'0.0);
   WRITEF(($2L,42A,5(L,-D.DD17QIQ-D.DD),L,28A,8(Q-D.DD)$,
           "COSINE COEFFICIENTS:    SINE COEFFICIENTS:",
           AB,
           "RESULTING DISCRETE FUNCTION:",
           'C06TRGSUM'AB))
"END'#OF 'C06TRGSUM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06TRGSUM' EXAMPLE PROGRAM.
DISCRETE FUNCTION, ON INPUT:  1.00 0.00 3.00 4.00
RESULTS:
COSINE COEFFICIENTS:    SINE COEFFICIENTS:
 4.00                 I  0.00
-1.00                 I -2.00
 0.00                 I  0.00
COSINE COEFFICIENTS:    SINE COEFFICIENTS: (ON INPUT)
 2.00                 I  0.00
 0.00                 I  1.00
 0.00                 I  2.00
 2.00                 I  3.00
 4.00                 I  0.00
RESULTING DISCRETE FUNCTION:  5.00 2.41 1.00 1.24 1.00 -0.41 5.00 -7.24
```

C06COSSUM and C06SINSUM

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06COSSUM' AND 'C06SINSUM' THIS
   PROGRAM CALCULATES APPROXIMATELY THE DISCRETE COSINE TRANSFORM AND
   THE DISCRETE SINE TRANSFORM.
   #
   'VEC'A=GENVEC(5):=(2.0,0.0,0.0,2.0,4.0);
   WRITEF(($28A,L,16A,5(Q-D.DD),L,36A,5(Q-D.DD)$,
           "'C06COSSUM' EXAMPLE PROGRAM.",
           "VECTOR ON INPUT:",
           A,
           "RESULT OF DISCRETE COSINE TRANSFORM:",
           'C06COSSUM'A));

   'VEC'B=GENVEC(5):=(0.0,1.0,2.0,3.0,0.0);
   WRITEF(($2L,28A,L,16A,5(Q-D.DD),L,34A,5(Q-D.DD)$,
            "'C06SINSUM' EXAMPLE PROGRAM.",
            "VECTOR ON INPUT:",
            B,
            "RESULT OF DISCRETE SINE TRANSFORM:",
            'C06SINSUM'B))
'END'#OF 'C06COSSUM' AND 'C06SINSUM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06COSSUM' EXAMPLE PROGRAM.
VECTOR ON INPUT:  2.00  0.00  0.00  2.00  4.00
RESULT OF DISCRETE COSINE TRANSFORM:  5.00  -2.41  3.00  0.41  1.00

'C06SINSUM' EXAMPLE PROGRAM.
VECTOR ON INPUT:  0.00  1.00  2.00  3.00  0.00
RESULT OF DISCRETE SINE TRANSFORM:  0.00  4.83  -2.00  0.83  0.00
```

14. Source texts

```
OP  C06TRGSUM =( COVEC ab) VEC :

#purpose:          approximately calculated is

                    m
                   sum" a[k+ LWB ab]×cos(pi×k×j/m)
                   k=0

                   +

                    m
                   sum b[k+ LWB ab]×sin(pi×k×j/m)
                   k=0

                   for j=0,1,...,2×m-1 with m= SIZE ab-1.
results:           the discrete harmonic synthesis i.e. the result of
                   the above formula for j=0,1, ... ,2×m-1 is
                   delivered as a  VEC  with bounds  LWB ab and
                   LWB ab+2×m-1. the vectors a and b must be given as
                   real and imaginary part of the  COVEC  ab.
exception handling:if m<1 then c06fail is called and the result is a
                   VEC  with bounds  LWB  ab and LWB ab+2×m-1.#

 IF  INT l= LWB ab,m= SIZE ab-1;m>0
 THEN  SCAL zero= WIDEN 0;
    IF im OF ab[1]≠zero OR im OF ab[l+m]≠zero
    THEN c06fail(2,
       "trgsumoperand of  C06TRGSUM  contains non-zero first and/or
       last element");
       im OF ab[1]:=im OF ab[l+m]:=zero
    FI ;
    ( C06EXPSUMHRM  CONJ ab)/<2
 ELSE c06fail(1,
    " TRGSUMOPERAND OF  c06trgsum  OF WRONG SIZE");
    genarray1(1,1+2×m-1)
 FI ,
```

```
OP  CO6TRGSUM =( VEC f) COVEC :

#purpose:          approximately calculated are

                              n-1
                   a[j]=1/m sum f[k+ LWB f]×cos(pi×k×j/m),
                              k=0

                   and

                              n-1
                   b[j]=1/m sum f[k+ LWB f]×sin(pi×k×j/m),
                              k=0

                   with n=2×m= SIZE f.
 input:            see above formula. f is not preserved.
 results:          the coefficients a and b are delivered as real and
                   imaginary part of a  COVEC  with bounds  LWB f and
                   LWB f+m.
exception handling:if n<1 then c06fail is called and the resulting
                   COVEC  has lower bound   lwbf and upper bound
                   LWB f+m. if n is odd then c06fail is called
                   and the smaller of the first and the last element
                   is discarded and  CO6TRGSUM  is called with the
                   modified operand.#

 IF  INT n= SIZE f,1= LWB f; INT m=n OVER 2;n>0
 THEN  IF  NOT  ODD n
    THEN ( CO6EXPSUM f)[1:1+m AT 1]/<m
    ELSE c06fail(3,
       "trgsumoperand (real vector) of  CO6TRGSUM  is of odd length");
       INT u= UPB f;
       IF  ABS f[1]< ABS f[u]
       THEN  CO6TRGSUM f[1+1:u AT 1]
       ELSE  CO6TRGSUM f[1:u-1 AT 1]
       FI
    FI
 ELSE c06fail(1,"trgsumoperand of  CO6TRGSUM OF WRONG SIZE");
    gencoarray1(1,1+m)
 FI ,
```

```
OP  C06COSSUM =( VEC a) VEC :
#purpose:          approximately calculated is
                    m
                   sum" a[k+ LWB a]×cos(pi×k×j/m), j=0,1, ... ,m
                   k=0

                   m= SIZE a-1.
input:             see above formula, a is not preserved.
results:           the cosine transform, i.e. the result of the
                   above formula is delivered as a  VEC  with bounds
                   similar to those of a.
exception handling:if m<1 then c06fail is called and the original
                   vector a is delivered.#

IF  INT m1= SIZE a;m1>1
THEN  C06EXPSUMHRM (a/<2)
ELSE c06fail(1,"cossumoperand of  C06COSSUM  of wrong size");a
FI ,



OP  C06SINSUM =( VEC a) VEC :
#purpose:          approximately calculated is

                    m
                   sum a[k+ LWB a]×sin(pi×k×j/m), j=0,1, ... ,m
                   k=0
                   m= SIZE a-1.
input:             see above formula, a is not preserved. note that:
                   the first and last element of a must be supplied
                   and filled with zero.
results:           the sine transform, i.e. the results of the
                   above formula is delivered as a  VEC  with bounds
                   similar to those of a. note that the first and last
                   element are zero again.
exception handling:if m<1 then c06fail is called and the original
                   vector a is delivered.#

IF  INT m1= SIZE a;m1>1
THEN  SCAL zero= WIDEN 0;
   IF a[ LWB a]≠zero OR a[ UPB a]≠zero
   THEN c06fail(2,
      "sinsumoperand of  C06SINSUM  containes nonzero first and/or
      last sine coefficient");
      a[ LWB a]:=a[ UPB a]:=zero
   FI ;
   C06EXPSUMANH (a/<2)
ELSE c06fail(1,"sinsumoperand of  C06SINSUM  of wrong size");a
FI
```

II.4 *C06TRGSUM, C06COSSUM, C06SINSUM* (Dyadic)

1. Purpose

The dyadic operators

C06TRGSUM, C06COSSUM, C06SINSUM

evaluate trigonometric sums.
Advantage has been taken of zeros in the data in

C06COSSUM - sine coefficients are zero
C06SINSUM - cosine coefficients are zero.

IMPORTANT:....

2. Specification (Algol 68)

```
MODE SCAL       = REAL, COSCAL = COMPL;
MODE VEC        = REF[ ]SCAL,
     COVEC      = REF[ ]COSCAL;
OP C06TRGSUM    = (SCAL t, COVEC ab) SCAL:
OP C06COSSUM    = (SCAL t, VEC a)    SCAL:
OP C06SINSUM    = (SCAL t, VEC b)    SCAL:
PRIO C06TRGSUM  = 8, C06COSSUM = 8, C06SINSUM = 8.
```

3. Description

The operators calculate

$$f(\theta, \underline{a}, \underline{b}) = \sum_{k=0}^{m} (a_k \cos k\theta + b_k \sin k\theta).$$

The algorithm used for evaluating a cosine sum and a sine sum is the Clenshaw algorithm with the modifications due to Reinsch.

4. References

[1] OLIVER , J.
An error analysis of the modified Clenshaw method for evaluating Chebyshev and Fourier series.
JIMA, vol. 20, 379-391. 1977.

5. Parameters

General:

- Both operands are preserved.
- The result is a real constant: the trigonometric sum.
- Only the size of the right operand matters: the k-th element of a vector a is assumed to be represented by *a[LWB a + k]*, so the lower bound of the data representation of the vector does not matter and is free for choice.
- m denotes the size of the coefficient vector minus 1.
- Left operand *t*: the angle θ; a real constant (it is advised to supply a value within $[-\pi,\pi)$).

Formula	Right operand
t C06TRGSUM ab	a complex array with m+1 elements: the real part contains the cosine coefficients, a, and the imaginary part contains the sine coefficients, b. The first element of b must contain zero.
t C06COSSUM a	a real array with m+1 elements: the coefficients of the cosine sum, a.
t C06SINSUM b	a real array with m+1 elements: the coefficients of the sine sum, b. The first element must contain zero.

6. Error indicators

In the event of an error condition being detected, the error routine: *c06fail* of mode *REF NAGFAIL*, is called with the parameters listed below. These are printed and in case the value of *c06fail* is *nagsoft* the execution is continued (see in Introduction of the NAG manual the document on the ALGOL 68 error mechanism). The operators were designed with *nagsoft* as the user-friendly error-handling mechanism in mind.

parameter	message
1	VECTOR OPERAND OF <*operator name*> OF WRONG SIZE The size of the given vector is smaller than one;

the result yielded is zero as value for the empty sum.

2 VECTOR OPERAND OF <*operator name*> CONTAINS NONZERO FIRST SINE COEFFICIENT

The calculation is performed with the first sine coefficient overwritten with zero.

7. Auxiliary routines None.

8. Timing

The time taken is proportional to m.

9. Storage No auxiliary arrays are declared.

10. Accuracy

Let us denote by:

. $\underline{a}$: the cosine coefficient vector;

. $\underline{b}$: the sine coefficient vector;

. θ : the angle;

. c_a, c_b : the condition numbers

$$c_a = \sum_{k=0}^{m} \max(1,|a_k|)/\max(1, |f(\theta,\underline{a},\underline{b})|),$$

$$c_b = \sum_{k=0}^{m} \max(1,|b_k|)/\max(1, |f(\theta,\underline{a},\underline{b})|);$$

. c_θ : the condition number

$$c_\theta = |\max(1,|\theta|)\frac{\partial f}{\partial \theta}|/\max(1, |f(\theta,\underline{a},\underline{b})|);$$

. g_a, g_b : growth factors (conjectured of order m);

. *a* : the machine representation of (the measured) $\underline{a}$;

. *b* : the machine representation of (the measured) $\underline{b}$;

. δa, δb : vectors of componentswise errors:

$$\delta a_k = |a_k - a[LWB\ a + k]|/\max(1, |a_k|)$$
$$\delta b_k = |b_k - b[LWB\ b + k]|/\max(1, |b_k|)$$

for all appropriate k;

. t : the machine representation of (the measured) θ;

. $\delta\theta$: $|t - \theta|/\max(1, |\theta|)$.

Error bounds (first order)

Formula	Propagated error	Generated error
t C06TRGSUM ab	$c_a * \|\underline{\delta a}\|_\infty + c_b * \|\delta b\|_\infty + c_\theta * \delta\theta$	$(c_a * g_a + c_b * g_b) *$ *small real*
t C06COSSUM a	$c_a * \|\underline{\delta a}\|_\infty + c_\theta * \delta\theta$	$c_a * g_a *$ *small real*
t C06SINSUM b	$c_b * \|\underline{\delta b}\|_\infty + c_\theta * \delta\theta$	$c_b * g_b *$ *small real*

11. Further comments None.

12. Keywords

Evaluation of trigonometric sum.

Clenshaw-Reinsch algorithm.

13. Examples

13.1 Program text.

```
'BEGIN'
   #
   AS AN ILLUSTRATION OF THE USE OF 'C06COSSUM', 'C06SINSUM' AND
   'C06TRGSUM' (DYADIC) THE PROGRAM CALCULATES APPROXIMATELY

    1
   SUM (A[K]*COS(K*T)+B[K]*SIN(K*T))
   K=0

   #
   'SCAL'ANGLE=PI/2;
   'COVEC'AB=(GENCOVEC(2):=(1.0'I'0.0,2.0'I'1.0))['AT'0];
   'VEC'A=RE'OF'AB,
        B=IM'OF'AB;
   WRITEF(($57A,L,6A,Q-D.DD,L,20A,2(Q-D.DD),L,18A,Q-D.DD,2L,18A,
              2(Q-D.DD),L,16A,Q-D.DD,2L,25A,Q-D.DD$,
          "'C06COSSUM', 'C06SINSUM' AND 'C06TRGSUM' EXAMPLE PROGRAM.",
          "ANGLE:",ANGLE,
          "COSINE COEFFICIENTS:",A,
          "RESULT COSINE SUM:",ANGLE'C06COSSUM'A,
          "SINE COEFFICIENTS:",B,
          "RESULT SINE SUM:",ANGLE'C06SINSUM'B,
          "RESULT TRIGONOMETRIC SUM:",ANGLE'C06TRGSUM'AB))
'END'#OF 'C06COSSUM', 'C06SINSUM' AND 'C06TRGSUM' EXAMPLE PROGRAM#
```

13.2 Data for program. None.

13.3 Results.

```
'C06COSSUM', 'C06SINSUM' AND 'C06TRGSUM' EXAMPLE PROGRAM.
ANGLE:  1.57
COSINE COEFFICIENTS:  1.00  2.00
RESULT COSINE SUM:  1.00

SINE COEFFICIENTS:  0.00  1.00
RESULT SINE SUM:  1.00

RESULT TRIGONOMETRIC SUM:  2.00
```

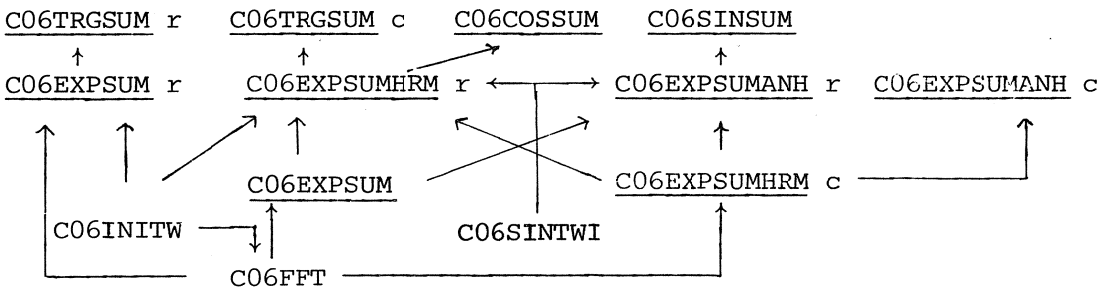
14. Source texts

```
OP  C06TRGSUM =( SCAL t, COVEC ab) SCAL :

#purpose:          approximately calculated is

                    m
                   sum (a[k]×cos(k×t)+b[k]×sin(k×t))
                   k=0

                   with m= SIZE ab-1.
input:             the angle t of  MODE  SCAL  must be given as left
                   operand. the right operand of  MODE  COVEC
                   contains the cosine coefficients as real part and
                   the sine coefficients as imaginary part. the first
                   element of the imaginary part is supposed to refer
                   to zero.
results:           the trigonometric sum of  MODE  SCAL .
exception handling:if m<0 then c06fail is called and zero is
                   delivered as the value of the empty  sum. if the
                   first element of the imaginary part does not refer
                   to zero then c06fail is called and the calculation
                   is performed after the element is overwritten with
                   zero.#

IF  SCAL zero= WIDEN 0; INT m= SIZE ab-1;m>-1
THEN  VEC a=re OF ab,b=im OF ab;
   IF b[ LWB b]≠zero
   THEN c06fail(2,
      "trgsumvector operand of  C06TRGSUM  contains nonzero first
         sine coefficient");
      b[ LWB b]:=zero
   FI ;
   SCAL sinsum,cossum;
   (c06ser(a,cossum, NIL ,t),c06ser(b, NIL ,sinsum,t));
   cossum+sinsum
ELSE c06fail(1,"trgsumvector operand of  C06TRGSUM  of wrong size");
   zero
FI ,
```

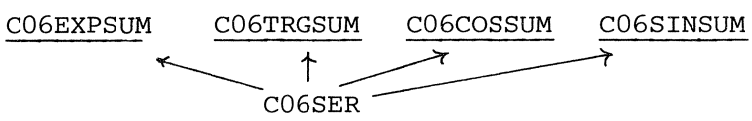
```
OP  C06COSSUM =( SCAL t, VEC a) SCAL :

#purpose:          approximately calculated is

                    m
                   sum a[ LWB a+k]×cos(k×t)
                   k=0

                   with m= SIZE a-1.
input:             the cosine coefficients must be given as a  VEC
                   with arbitrary first index.
result:            the cosine sum.
exception handling:if m<0 then c06fail is called and zero is
                   delivered for the empty sum.#

IF  INT m= SIZE a-1;m>-1
THEN  SCAL cossum;
   c06ser(a,cossum, NIL ,t);cossum
ELSE c06fail(1,"cossumvector operand of  C06COSSUM  of wrong size");
   WIDEN 0
FI ,



OP  C06SINSUM =( SCAL t, VEC b) SCAL :

#purpose:          approximately calculated is

                    m
                   sum b[ LWB b+k]×sin(k×t)
                   k=0

                   with m= SIZE b-1.
input:             the sine coefficients must be given as a  VEC
                   with arbitrary first index. the first element is
                   supposed to refer to zero.
result:            the sine sum.
exception handling:if m<0 then c06fail is called and zero is
                   delivered as the value of the empty sum. if
                   b[ LWB b] does not refer to zero then c06fail is
                   called and the calculation is performed after
                   b[ LWB b] is overwritten with zero.#

IF  SCAL zero= WIDEN 0; INT m= SIZE b-1;m>-1
THEN  IF b[ LWB b]≠zero
   THEN c06fail(2,
      "sinsumvector operand of  C06SINSUM  contains nonzero first
         sine coefficient");
      b[ LWB b]:=zero
   FI ;
   SCAL sinsum;
   c06ser(b, NIL ,sinsum,t);
   sinsum
ELSE c06fail(1,"sinsumvector operand of  C06SINSUM  of wrong size");
   zero
FI
```

III. SOURCE TEXTS OF THE TECHNICAL ROUTINES

III.1 Hierarchy of the implementations (monadic operators)



III.2 Hierarchy of the implementations (dyadic operators)



III.3 Source texts

```
PROC  c06sintwi=( INT n, VEC s) VOID :

#purpose:the points sin(2×pi×j/n), j=0,1, ... ,  UPB s
         are calculated.
input:    VEC s  with  LWB s=0 and  UPB s>0;
         INT n  with n>0.
results: the points are delivered in s.#

IF  INT u= UPB s; LWB s=0  AND  u>0  AND  n>1
THEN  IF u<n
   THEN s[0]:= WIDEN 0;
      IF u>0
      THEN  SCAL spi=pi,two= WIDEN 2;
         SCAL  tpi=two×spi;
         SCAL the=tpi/ WIDEN n;
         s[1]:=sin(the);
         INT ttp:=1; INT uu=u MIN (n OVER 2);
         WHILE ttp×:=2;ttp<uu
         DO s[ttp]:=sin(ttp×the) OD ;
         INT thp=ttp OVERAB 2;
         IF  INT l:=thp;l<uu
         THEN  WHILE (ttp OVERAB 2)>0
            DO  IF l+ttp<uu
               THEN l+:=ttp;s[l]:=sin(l×the)
               FI
            OD
         FI ;
         INT k:=thp OVER 2;
         WHILE  (k  OVERAB  2) > 1
         DO  IF  SCAL tct=two×cos(k×the); ABS tct> WIDEN 1
            THEN  FOR  j  FROM  3×k  BY  2×k  TO  uu-k
               DO s[j]:=(s[j-k]+s[j+k])/tct OD
            ELSE  FOR  j  FROM  3×k  BY  2×k  TO  uu-k
               DO s[j]:=tct×s[j-k]-s[j-2×k] OD
            FI
         OD ;
         FOR j FROM uu+1 TO u DO s[j]:=-s[n-j] OD
      FI
   ELSE c06sintwi(n,s[0:n-1 AT 0]);
     FOR j FROM n TO u DO s[j]:=s[j-n] OD
   FI
ELSE  c06fail(1,"sintwiwrong size and/or bounds in c06sintwi")  FI ,
```

```
PROC  c06initw=( INT n, COVEC w) VOID :

#purpose:the twiddle factors exp(0 I 2×pi×j/n), j=0,1, ... ,  UPB w
         are calculated.
input:    COVEC w  with  LWB w=0 and  UPB w>0;
         INT n  with n>0.
results: the twiddle factors are delivered in w.#

IF  INT u= UPB w; LWB w=0  AND  u>0  AND  n>1
THEN  IF u<n
   THEN  SCAL scalpi=pi,two= WIDEN 2,one= WIDEN 1;
      SCAL  tpi=scalpi+scalpi;
      SCAL  kth:=(n=1! WIDEN 0!tpi/ WIDEN n),
      INT k:=1;
      INT uu=u MIN (n OVER 2);
      w[0]:= WIDEN one;
      WHILE k <uu
      DO w[k]:=(cos(kth),sin(kth));
         k×:=2;
         kth×:=two
      OD ;
      INT i=k OVERAB 2;k OVERAB 2;
      WHILE  (k  OVERAB  2) > 1
      DO  IF  SCAL  tct=two×re OF w[k]; ABS  tct >  one
         THEN  FOR  j  FROM  3×k  BY  2×k  TO  i-k
            DO w[j]:=(w[j-k]+w[j+k])/tct OD
         ELSE  FOR  j  FROM  3×k  BY  2×k  TO  i-k
            DO w[j]:=tct×w[j-k]-w[j-2×k] OD
         FI
      OD ;
      FOR j FROM i+1 TO uu DO w[j]:=w[i]×w[j-i] OD ;
      FOR j FROM uu+1 TO u DO w[j]:= CONJ w[n-j] OD
   ELSE c06initw(n,w[0:n-1 AT 0]);
      FOR j FROM n TO u DO w[j]:=w[j-n] OD
   FI
ELSE  c06fail(1,"c06initw wrong size and/or bounds")
FI ,
```

```
PROC  c06fft=( COVEC xy) VOID :

#purpose:          approximately calculated is

                   n-1
                   sum w[j,k]×xy[k+ LWB xy],   j=0,1, ... ,n-1
                   k=0

                   with n= SIZE xy,  w[j,k]=exp(0 I j×k×2×pi/n).
input:             see above formula.
results:           the idft of xy is delivered with
                   bounds pair 0:n-1.
exception handling:if  NOT ( LWB xy=0 AND  SIZE xy>0)then
                   c06fail is called.#

IF  INT n= UPB xy+1;  LWB xy=0  AND  n>1  THEN
   IF  n>1  THEN

PROC fftri=( COVEC x,w, INT ri,rlriminl,ripluslrm,n) VOID :
CASE
   PROC xi=( INT i) COVEC :x[ AT -i×ripluslrm],
   COSCAL zero= WIDEN  WIDEN 0,
   INT rirm=ri×ripluslrm,noverri=n OVER ri,risubl=ri-1,
       ripluslrmsubl=ripluslrm-1;
   INT nsubrirm=n-rirm;
   risubl
IN
BEGIN
   COVEC x0=xi(0),x1=xi(1);
   FOR  k  FROM  0  TO  ripluslrmsubl  DO
      COSCAL  wn=w[k×rlriminl];
      FOR  j  FROM  k  BY  rirm  TO  nsubrirm+k  DO
         REF  COSCAL x0j=x0[j],x1j=x1[j];
         COSCAL a=x0j,b=x1j;
         (x0j:=a+b,x1j:=(a-b)×wn)
      OD
   OD
END ,
BEGIN
   COVEC x0=xi(0),x1=xi(1),x2=xi(2);
   COSCAL  ei120=w[noverri],ei240=w[2×noverri];
   FOR  k  FROM  0  TO  ripluslrmsubl  DO
      INT ind=k×rlriminl;
      COSCAL  w1n=w[ind],w2n=w[2×ind];
      FOR  j  FROM  k  BY  rirm  TO  nsubrirm+k  DO
         REF  COSCAL x0j=x0[j],x1j=x1[j],x2j=x2[j];
         COSCAL a=x0j,b=x1j,c=x2j;
         (x0j:=(a+b      +c      )    ,
          x1j:=(a+b×ei120+c×ei240)×w1n,
          x2j:=(a+b×ei240+c×ei120)×w2n)
      OD
   OD
END ,
```

```
BEGIN
   COVEC x0=xi(0),x1=xi(1),x2=xi(2),x3=xi(3),
   PROC ( COSCAL ) COSCAL ei90=
          ( COSCAL a) COSCAL :(- IM a, RE a) ;
   IF  noverri=rlriminl  THEN
      FOR  j  FROM  0  BY  rirm  TO  nsubrirm    DO
         REF  COSCAL x0j=x0[j],x1j=x1[j],x2j=x2[j],x3j=x3[j];
         COSCAL e=x0j+x2j,f=x0j-x2j,g=x1j+x3j,h=ei90(x1j-x3j);
         (x0j:=e+g,x2j:=e-g,x1j:=f+h,x3j:=f-h)
      OD
   ELSE
      FOR  k  FROM  0  TO  riplusrmsubl  DO
         INT ind=k×rlriminl;
         COSCAL  w1n=w[ind],w2n=w[2×ind],w3n=w[3×ind];
         FOR  j  FROM  k  BY  rirm  TO  nsubrirm+k  DO
            REF  COSCAL x0j=x0[j],x1j=x1[j],x2j=x2[j],x3j=x3[j];
            COSCAL e=x0j+x2j,f=x0j-x2j,g=x1j+x3j,h=ei90(x1j-x3j);
            (x0j:=e+g,x2j:=(e-g)×w2n,x1j:=(f+h)×w1n,x3j:=(f-h)×w3n)
         OD
      OD
   FI
END ,
BEGIN
   COVEC x0=xi(0),x1=xi(1),x2=xi(2),x3=xi(3),x4=xi(4);
   COSCAL  ei 72=w[  noverri],ei144=w[2×noverri],
           ei216=w[3×noverri],ei288=w[4×noverri];
   FOR  k  FROM  0  TO  riplusrmsubl  DO
      INT ind=k×rlriminl;
      COSCAL  w1n=w[ind],w2n=w[2×ind],w3n=w[3×ind],w4n=w[4×ind];
      FOR  j  FROM  k  BY  rirm  TO  nsubrirm+k  DO
         REF  COSCAL x0j=x0[j],x1j=x1[j],x2j=x2[j],
                     x3j=x3[j],x4j=x4[j];
         COSCAL a=x0j,b=x1j,c=x2j,d=x3j,e=x4j;
         (x0j:=(a+b      +c      +d      +e      )    ,
          x1j:=(a+b×ei 72+c×ei144+d×ei216+e×ei288)×w1n,
          x2j:=(a+b×ei144+c×ei288+d×ei 72+e×ei216)×w2n,
          x3j:=(a+b×ei216+c×ei 72+d×ei288+e×ei144)×w3n,
          x4j:=(a+b×ei288+c×ei216+d×ei144+e×ei 72)×w4n)
      OD
   OD
END
OUT
IF  ri=n  THEN
   COVEC xp= COPY (x);
   BEGIN
      REF  COSCAL s=x[0]:=zero;
      FOR  p  FROM  0  TO  risubl  DO  s+:=xp[p]  OD
   END ;
   FOR  j  FROM  1  TO  risubl  DO
      INT  b:=j,
      REF  COSCAL s=x[j]:=xp[0];
      FOR  p  FROM  1  TO  risubl  DO
         s+:=xp[p]×w[b];
         ( (b+:=j) > n ! b-:=n )
      OD
   OD
```

```
 ELSE
    [0:risubl] COVEC  xp;
    FOR  p  FROM  0  TO  risubl  DO  xp[p]:=xi(p)  OD ;
    FOR  k  FROM  0  TO  ripluslrmsubl  DO
       INT ind=k×rlriminl;
       FOR  j  FROM  k  BY  rirm  TO  nsubrirm+k  DO
          COVEC xpj=gencoarrayl(0,risubl);
          BEGIN
             COSCAL  s:=zero;
             FOR  p  FROM  0  TO  risubl  DO
                s+:=(xpj[p]:=xp[p][j])
             OD ;
             xp[0][j]:=s
          END ;
          INT c:=0,d:=0;
          FOR  p  FROM  1  TO  risubl  DO
             (c+:=noverri,d+:=ind);
             INT b:=d,
             REF  COSCAL s=xp[p][j]:=zero;
             FOR  q  FROM  0  TO  risubl  DO
                s+:=xpj[q]×w[b];
                ((b+:=c)>n!b-:=n)
             OD
          OD
       OD
    OD
 FI
 ESAC ,

 MODE  R = STRUCT ( INT ri,rlriminl,ripluslrm),
 MODE  L = STRUCT ( REF  R r, REF  L next),

 PROC factor=( INT n, REF  REF  L l13,l2,l123, REF  INT rimax) VOID :
 BEGIN
    REF  L klb:= NIL ,kle,k2b:= NIL ,k2e,k3b:= NIL ,k3e,
            mlb:= NIL ,mle,m2b:= NIL ,m2e,m3b:= NIL ,m3e,
    PROC inlist=( INT ri) VOID :
    IF
       (ri>rimax!rimax:=ri);
       PROC list=( REF  REF  L lb,le, BOOL bef, REF  R r) VOID :
       IF  lb:=: REF  L ( NIL )  THEN
          lb:=le:= HEAP  L :=(r, NIL )
       ELIF  bef  THEN
          lb:= HEAP  L :=(r,lb)
       ELSE
          le:=next OF le:= HEAP  L :=(r, NIL )
       FI ;
       lastri=0
    THEN
       lastri:=ri
    ELIF  lastri=ri  THEN
```

```
    HEAP  R r1,r2;
    ri OF r1:=ri OF r2:=ri;
    list(k1b,k1e, FALSE ,r1);list(m1b,m1e, FALSE ,r1);
    list(k3b,k3e, TRUE  ,r2);list(m3b,m3e, TRUE  ,r2);
    lastri:=0
 ELSE
    HEAP  R r;
    ri OF r:=lastri;
    list(k2b,k2e, FALSE ,r );list(m2b,m2e, FALSE ,r );
    lastri:=ri
 FI ,
    INT npart:=n,lastri:=0;
    rimax:=0;

    WHILE npart  MOD  4=0  DO
       npart  OVERAB  4;inlist(4)
    OD ;
    IF  NOT  ODD  npart  THEN
       npart  OVERAB  2;inlist(2)
    FI ;
    INT div:=3;
    WHILE
       IF  npart  MOD  div =0  THEN
          npart  OVERAB  div;inlist(div);
          TRUE
       ELIF  npart  OVER  div>div  THEN
          div+:=2;
          TRUE
       ELSE  FALSE
       FI
    DO  SKIP  OD ;
    IF  npart>1  THEN  inlist(npart)  FI ;
    inlist(0);

    IF  k1b:=: REF  L ( NIL ) THEN
       113:= NIL ;12:=1123:=k2b
    ELIF  k2b:=: REF  L ( NIL )  THEN
       12:= NIL ;113:=1123:=k1b;next OF k1e:=k3b
    ELSE
       12:=k2b;113:=k1b;next OF k1e:=k3b;
       1123:=m1b;next OF m1e:=m2b;next OF m2e:=m3b
    FI ;

    INT r1riminl:=1,
    REF  L l:=1123;
    WHILE
       REF  R r=r OF l;
       r1riminl  OF r:=r1riminl ;
       ripluslrm OF r:=n OVER (r1riminl×:=ri OF r);
       (l:=next OF l):≠: REF  L ( NIL )
    DO  SKIP  OD
END ,
```

```
PROC  revers=( COVEC x, REF  L 113,12, INT n) VOID :
IF
   PROC  perm=( REF  L 1, PROC ( INT , INT ) VOID pr) VOID :
   BEGIN
      PROC pp=( REF  L 1, INT j,k) VOID :
      IF  REF  R r=r OF 1;next OF 1:=: REF  L ( NIL )  THEN
         FOR  p  FROM  0  TO  ri OF r-1  DO
            pr(j+p×rlriminl OF r,k+p×ripluslrm OF r)
         OD
      ELSE
         FOR  p  FROM  0  TO  ri OF r-1  DO
            pp(next OF 1,j+p×rlriminl OF r,k+p×ripluslrm OF r)
         OD
      FI ;
      pp(1,0,0)
   END ;

      (113:≠: NIL )  AND  (12: =: NIL )  THEN
   perm(113,( INT j,k) VOID :(j<k!x[j]=:=x[k]))
ELIF (113: =: NIL )  AND  (12:≠: NIL )  THEN
   IF  next OF 12:≠: REF  L ( NIL )  THEN
      INDEX p=genintarray(0,n-1);
      perm(12,( INT j,k) VOID :p[j]:=k);
      FOR  j  FROM  1  TO  n-2  DO
         IF  INT  k:=p[j]; j≠k  THEN
            COSCAL s=x[k];p[j]:=j;
            WHILE  INT l=p[k];l≠k  DO
               x[k]:=x[l];p[k]:=k;k:=l
            OD ;
            x[k]:=s
         FI
      OD
FI
ELIF (113:≠: NIL )  AND  (12:≠: NIL )  THEN
   IF  next OF 12: =: REF  L ( NIL )  THEN
      INT step=ripluslrm OF r OF 12;
      INT stepspan=step×(ri OF r OF 12-1);
      PROC  change=( INT j,k) VOID :
      IF  j <  k  THEN
         COVEC xj=x[ AT -j],xk=x[ AT -k];
         FOR  p  FROM  0  BY  step  TO  stepspan  DO
            xj[p]=:=xk[p]
         OD
      FI ;
      perm(113,change)
   ELSE
      INT step=( REF  L l:=12;  WHILE next OF l:≠: REF  L ( NIL )
                 DO  l:=next OF l  OD ;ripluslrm OF r OF l);
      INT span=n OVER (step×rlriminl OF r OF 12)-1;
      MODE  CYCLE = STRUCT ( INT no, REF  CYCLE next),
      [0:span] CYCLE p;
      perm(12,( INT j,k) VOID :p[j OVER step]:=(k OVER step, NIL ));
```

```
MODE  LIST = STRUCT ( REF  CYCLE start, REF  LIST next),
REF  LIST l:= NIL ,
INT j:=0;
WHILE
   IF  REF  CYCLE  t:=p[j];
        next  OF  t :=:  REF  CYCLE ( NIL )  THEN
      WHILE
         REF  CYCLE s=t; INT k=no OF s;
         t:=p[k];s:=(j×step,t); j:=k;
         next OF t :=:  REF  CYCLE ( NIL )
      DO  SKIP  OD ;
      l:= HEAP  LIST :=(t,l)
   FI ;
   j <  span
DO  j+:=1  OD ;

PROC  listperm=( INT j,k) VOID :
IF  j=k  THEN
   REF  LIST list:=l,
   COVEC xj=x[ AT -j];
   WHILE
      IF  REF  CYCLE start=start OF list;
           next  OF  start :≠: start  THEN
         REF  CYCLE  t:=start, INT no:=no OF start;
         COSCAL s=xj[no];
         WHILE
            INT n=no OF (t:=next OF t);
            xj[no]:=xj[n];no:=n;
            next  OF  t :≠: start
         DO  SKIP  OD ;
         xj[no]:=s
      FI ;
      (list:=next OF list):≠: REF  LIST ( NIL )
   DO  SKIP  OD
ELIF  j<k  THEN
REF  LIST list:=l,
COVEC xj=x[ AT -j],xk=x[ AT -k];
WHILE
   IF  REF  CYCLE start=start OF list;
        next  OF  start  :=: start  THEN
      INT no=no OF start;
      xj[no]=:=xk[no]
   ELSE
      REF  CYCLE  t:=start, INT no:=no OF start;
      COSCAL sl=xj[no],s2=xk[no];
      WHILE
         INT n=no OF (t:=next OF t);
         (xj[no]:=xk[n],xk[no]:=xj[n]);no:=n;
         next  OF  t :≠: start
      DO  SKIP  OD ;
      (xk[no]:=sl,xj[no]:=s2)
   FI ;
```

```
            (list:=next OF list):≠: REF  LIST ( NIL )
         DO  SKIP  OD
      FI ;

      perm(113,listperm)
   FI
FI ,

      INT rimax,
      REF  L 113,12,1123;
      factor(n,113,12,1123,rimax);
      COVEC  w=gencoarrayl(0,n-(rimax>5!1!n OVER rimax) );
      c06initw(n,w);
      WHILE
         REF  R r=r OF 1123;
         fftri(xy,w,ri OF r,rlriminl OF r,ripluslrm OF r,n);
         (1123:=next OF 1123):≠: REF  L ( NIL )
      DO  SKIP  OD ;
      revers(xy,113,12,n)
   FI
ELSE  c06fail(1,"c06fft wrong size")  FI ,
```

```
PROC c06ser=( VEC a, REF  SCAL cosser,sinser, SCAL t) VOID :

#purpose: calculation of either or both of the trigonometric sums
     u
   sum aa[k]×cos(k×t)          (cosine sum)
   k=0

     u
   sum aa[k]×sin(k×t)          (sine sum  )
   k=1

   with aa[k]=a[ LWB a+k]  and  u= SIZE a-1.

input parameters:  SCAL t  angle of trigonometric sum,
                   VEC a  coefficients of trigonometric sum,
                   REF  SCAL cosser,sinser  when containing  NIL  no
                      cosine and/or sine sum are desired otherwise
                      the cosine and/or the sine sum are desired.
output parameters:  REF  SCAL cosser,sinser  they will contain
                      the cosine and sine sum, provided they
                      were not pointing to  NIL  on input.#

IF (cosser:=: NIL ! SIZE a>0! SIZE a>0)   THEN
   PROC sqr=( SCAL c) SCAL :c×c,
   SCAL c=cos(t),zero= WIDEN 0,
   one= WIDEN 1,two= WIDEN 2,four= WIDEN 4;
   SCAL half=one/two;
   INT ua= UPB a,lapl= LWB a+1;
   IF  c<-half  THEN
      SCAL lambda=four×sqr(cos(t/two)), SCAL un:=zero,dun:=zero;
      FOR  k  FROM  ua  BY  -1  TO  lapl
         DO  dun:=lambda×un-dun+a[k];un:=dun-un  OD ;
      IF  cosser:≠: NIL  THEN  cosser:=lambda/two×un-dun+a[lapl-1]
      FI ;
      IF  sinser:≠: NIL  THEN  sinser:=un ×sin(t)          FI
   ELIF  c>half  THEN
     SCAL lambda=-four×sqr(sin(t/two)), SCAL un:=zero,dun:=zero;
      FOR  k  FROM  ua  BY  -1  TO  lapl
         DO  dun:=lambda×un+dun+a[k];un:=dun+un  OD ;
      IF  cosser:≠: NIL  THEN  cosser:=lambda/two×un+dun+a[lapl-1]
      FI ;
      IF  sinser:≠: NIL  THEN  sinser:=un ×sin(t)          FI
   ELSE
      SCAL  cc=c+c , SCAL  un1:=zero,un2:=zero;
      FOR  k  FROM  ua  BY  -1  TO  lapl
         DO   SCAL  h=cc×un1-un2+a[k];un2:=un1;un1:=h  OD ;
      IF  cosser:≠: NIL  THEN  cosser:=un1×c-un2+a[lapl i]
      FI ;
      IF  sinser:≠: NIL  THEN  sinser:=un1×sin(t)          FI
   FI
ELSE  c06fail(1,"c06ser wrong size")  FI
```

IV. TESTING

Apart from the example test programs given in the documentation units we considered for the stringent tests the cases:

. problems with known exact results (model problem),

. verification of relation of Parseval,

. verification of the pair: transformation and its inverse.

The dyadic operators are applied to those argument values implicit in the monadic operators. The resulting formulas of the above cases were published (in Dutch) in MC Syllabus 29.1b p. 227-231.

V . FUTURE PLANS

For the near future implementations are considered for

. general summation (V.1)

. summation of Chebyshev sums (V.2)

. summation of sums of orthogonal polynomials (V.3)

. two-dimensional IDFT (V.4)

. operators for special matrix-times-vector products (V.5)

. Winograd technique for the improvement of the DFT (V.6)

Anyone who likes to contribute with respect to the above items - or has suggestions with respect to any other item within the C6 chapter - is encouraged to contact the author.

When appropriate we refer to the NUMAL library of the Mathematical Centre for ALGOL 60 implementations.

V.1 General summation

Within this context we have for problem (1.1)

$$\sum_{k=\ell}^{\infty} f_k .$$

For a slowly convergent series an Euler transformation with van Wijngaarden strategy can be used; when the terms of the series have the same sign a preliminary transformation (due to Van Wijngaarden) can be applied to transform the series into an alternating one. Apart from those linear transformations a lot of non-linear techniques are available. For a recent survey see Brezinski (1978) with implementations as FORTRAN programs. It requires more research in order to make a more detailed proposal.

Remarks

. A routine Euler is provided in the RR of ALGOL 68.
. In NUMAL implementations are available for an alternating series and for a series with terms of the same sign.

Literature

Brezinski, C. (1978): Algorithmes d'accélération de la convergence. Étude numérique, Paris.
Daniel, J.W. (1969): Summation of a series of positive terms by condensation transformations. Math. Comp., 23, 91-96.
Van Wijngaarden, A. (1965): Course Scientific computing B; process analysis (Dutch). Mathematisch Centrum CR-18.

V.2 The summation of Chebyshev sums

Within this context we have for problem (1.1)

$$S(x) = \sum_{k=0}^{n} a_k T_k(x), \qquad x \in [-1,1]$$

with $T_k(x)$ the Chebyshev polynomial of the first kind of degree k.
A well-known algorithm for the evaluation of this sum is the Clenshaw algorithm, which can easily be understood from

$$S(x) = (1,-x) \sum_{k=0}^{n} \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} a_k \\ 0 \end{pmatrix}$$

by applying Horner's rule to the matrix polynomial.

As a special case we have the odd Chebyshev sum

$$S_o(x) = \sum_{k=0}^{n} a_k T_{2k+1}(x).$$

The Clenshaw algorithm for the evaluation of the above sum is easily obtained from

$$S_o(x) = x(1,-1) \sum_{k=0}^{n} \begin{pmatrix} 2T_2(x) & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} a_k \\ 0 \end{pmatrix}$$

by again applying Horner's rule to the matrix polynomial.

The even Chebyshev sum

$$S_e(x) = \sum_{k=0}^{n} a_k T_{2k}(x)$$

can be reduced to the calculation of S because

$$T_{2k}(x) = T_k(T_2(x))$$

and therefore

$$S_e(x) = S(T_2(x)).$$

The summation of the shifted Chebyshev sum

$$S^*(x) = \sum a_k T_k^*(x), \qquad x \in [0,1]$$

can be reduced to the summation of S because

$$T_k^*(x) = T_k(2x-1),$$

and therefore

$$S^*(x) = S(2x-1).$$

For the above problems it is proposed to implement the dyadic operators with left operand scal x and right operand vec a:

C06CHESUM for $S(x)$
C06ODDCHESUM for $S_o(x)$
C06EVECHESUM for $S_e(x)$
C06SHTCHESUM for $S^*(x)$.

Remarks.

. Implementations in ALGOL 60 are provided in NUMAL.
. We agree with the remark of Curtiss that the modifications due to Reinsch of the Clenshaw-algorithm are only useful, if we have ϕ available instead of x, with $x = \cos \phi$. The implementation of Cox and Hayes (1974) is therefore not necessary.
. It has been observed by Newbery (1974), that if the coefficients of the equivalent power sum representation are of the same sign or strictly alternating, then the power sum representation can be used instead of the Chebyshev sum representation, without loss of accuracy and with gain in evaluation speed. If we define the sensitivity for the perturbations in the coefficients $\{a_k\}$ of $S(x)$ by

$$K(S_a(x)) = \sum_k \left| \frac{a_k}{S} \frac{\partial S(x)}{\partial a_k} \right|$$

and $K(S_b)$ analogous for S in the Chebyshev sum representation, then we have under the conditions mentioned by Newbery:

$$\max_{x\in[-1,1]} K(S_a(x)) = \max_{x\in[-1,1]} K(S_b(x)).$$

(An implementation for the transformation of a power sum into a Chebyshev sum or vice versa is provided in NUMAL).

Literature.

Cox, M.G. & J.G. Hayes (1974): Curve fitting: a guide and suite of algorithms for the non-specialist user. NAC Report 26. National Physical Laboratory.

Newbery, A.C.R. (1974): Error analysis for polynomial evaluation. Math. Comp., 28, 789-793.

V.3 The summation of sums of orthogonal polynomials

Within this context we have for problem (1.1)

$$S(x) = \sum_{k=0}^{n} a_k f_k(x)$$

with $f_k(x)$ a function obeying a second order homogeneous recurrence relation. As an important special case we have the summation of orthogonal polynomials. The names of classical orthogonal polynomials and some of their properties and interrelations are given in chapter 22 of Abramowitz and Stegun. Correlated with the names is the standardization. In our opinion it is advantageous to provide, at first, implementations for:

a. a sum of orthogonal polynomials each normalized with the coefficient of the highest power of x equals 1;
b. sums of diverse orthogonal polynomials, named and normalised according to Abramowitz and Stegun.

The implementation under a) is general in the sense, that the appropriate recurrence coefficients, besides the argument x and coefficients, $\{a_k\}$, must be provided by the user; the implementations under b) are recognised by their names and only the argument, the coefficients $\{a_k\}$ and appropriate parameters must be provided by the user.

The algorithms are essential due to Clenshaw, because, if

$$f_{k+1} = \alpha_k(x) f_k + \beta_k f_{k-1}, \qquad k = 1,2,\ldots$$

with initial values f_0 and f_1, are given, then

$$S(x) = (f_0, f_1) \sum_{k=0}^{n} \prod_{j=0}^{k-1} \begin{pmatrix} \alpha_j(x) & 1 \\ \beta_j & 0 \end{pmatrix} \begin{pmatrix} a_k \\ 0 \end{pmatrix}$$

is easily obtained from the Horner-like rule

$$(f_0, f_1)\left(\begin{pmatrix} a_0 \\ 0 \end{pmatrix} + A_0\left(\begin{pmatrix} a_1 \\ 0 \end{pmatrix} + \ldots + A_{n-2}\left(\begin{pmatrix} a_{n-1} \\ 0 \end{pmatrix} + A_{n-1}\begin{pmatrix} a_n \\ 0 \end{pmatrix}\right)\ldots\right)\right)$$

with

$$A_j = \begin{pmatrix} \alpha_j(x) & 1 \\ \beta_j & 0 \end{pmatrix} .$$

This 'Horner-scheme' can be viewed as an inhomogeneous second order recursion.

For the implementation under a) we propose the dyadic operators

name	left operand	right operand
C06SUMORTPOL	struct (vec b,c, scal x)	vec a
C06SUMORTPOL	struct (vec c, scal x)	vec a

The recurrence coefficients b,c are related to those given in table 22.7 of Abramowitz and Stegun by

$$b_k = -a2_k/a3_k \quad , \qquad k = 0,1,..$$

$$c_k = a4_k * a1_{k-1}/(a3_k * a3_{k-1}), \qquad k = 1,2,... \ .$$

For the well-known classical orthogonal polynomials we have

polynomial kind	recurrence coefficients	
	b_k	c_k
Chebyshev (1st kind)	0	1/2, k = 1 1/4, k > 1
Chebyshev (2nd kind)	0	1/4
Legendre	0	$k^2/(4k^2-1)$
Jacobi	$\frac{(\beta^2-\alpha^2)}{(\alpha+\beta+2k)(\alpha+\beta+2(k+1)}$	$\frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+2k)^2(\alpha+\beta+3)}$, k = 1 $\frac{4k(\alpha+k)(\beta+k)(\alpha+\beta+k)}{(\alpha+\beta+2k)^2((\alpha+\beta+2k)^2-1)}$,k>1
Laguerre	$\alpha + 2k + 1$	$(\alpha+k)k$
Hermite	0	$k/2$

For the sum S(x) we have

$$S(x) = \sum_{k=0}^{n} a_k f_k(x) = (b,c,x)\ \underline{C06SUMORTPOL}\ a'$$

with $f_k(x)$ as defined by Abramowitz and Stegun in table 22.7 and

$$a'_k = \prod_{j=0}^{k-1} (a3_j/a1_j)a_k, \qquad k = 0,1,\ldots n.$$

Remarks.

- The summation of polynomials with $b_k = 0$, is catered for in a simple fashion.
- In NUMAL an implementation, heavily based on Gautschi (1968), is provided for obtaining the recurrence coefficients of a general orthogonal polynomial with a positive weight function.
- The used techniques may be applied to the summation of functions, which obey a three-term recurrence relation. However, the numerical stability must be ascertained for every particular case: if either of the solutions of the homogeneous recurrence dominate the solution of the inhomogeneous recurrence, then the problem is unstable, and a modification of the problem by eliminating the dominant solution is necessary; a criterium in terms of the eigenvalues of the matrices A_j and the coefficients $\underline{a}$ is not yet provided.

For the implementations under b) we propose the dyadic operators with right operand vec a and

name	left operand	$f_k(x)$
C06JACSUM	struct (scal α,β,x)	$P_k^{(\alpha,\beta)}$
C06GEGSUM	struct (scal α,x)	$C_k^{(\alpha)}$
C06CHESUM		(see previous paragraph)
C06TSJSUM	scal x	U_k
C06LEGSUM	scal x	P_k
C06LAGSUM	struct (scal α,x)	$L_k^{(\alpha)}$
C06HERSUM	scal x	H_k

Literature

Gautschi, W. (1968): Algorithm 331, Gaussian Quadrature formulas, Comm. ACM, 11.

V.4 Two dimensional IDFT

The computational problem is

$$\sum_{\ell=0}^{n-1} \sum_{k=0}^{n-1} a_{k\ell}\, W_n^{kp}\, W_n^{\ell q}, \qquad p,q = 0,1,\ldots,n-1$$

which can be dealt with:

```
for q to n
do   C06EXPSUM a[  ,q]od;
for p to n
do   C06EXPSUM a[p, ]od
```

where we assumed

$$a_{k\ell} = a[1+k,1+\ell].$$

Remarks.

- Henrici (1979;see introduction) mentions, that application of the DFT-idea direct to a multi-dimensional structure is more efficient. For the two-dimensional case it is not clear, whether it is worth the possible more complicated and more time-consuming bookkeeping. Within the context of the applications an improvement of 10 to 20% in speed is worthwhile.
- Various authors mention the storage problems for large n.
- Two dimensional Fourier transforms with data provided equidistant in the the r,ϕ-plane are desired.

V.5 Operators for special matrix-times-vector products

In this paragraph we consider representations of circulant and Toeplitz matrices.

These representations are used to form matrix-time-vector products fast; and when we can handle the multiplication fast, it is worthwhile to consider the solution of linear equations with Toeplitz matrices. So far we have not found DFT methods for solving general Toeplitz systems of equations although we are aware of the work of Trench (Zohar (1969)), Zohar (1974), Farder (1977), Kailath T. c.s. (1978), and de Meersman (1975).

5.1 Conventions, notations and relations

. We restrict ourselves to n*n-matrices

. Circulant matrix

$$C(a) = \begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix} \quad \text{,with } a = \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix}$$

. Toeplitz matrix

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix} \quad \text{,with } a = \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix}$$

$$a_{-} = \begin{pmatrix} a_0 \\ a_{-1} \\ \cdot \\ \cdot \\ \cdot \\ a_{-(n-1)} \end{pmatrix}.$$

. Hankel matrix

$$H(a) = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{n-1} & \cdots & & a_{2n-1} \end{pmatrix}$$

. upper triangular Toeplitz matrix

$$\nabla(a) = \begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ & a_0 & \ddots & \vdots \\ & & \ddots & a_{n-1} \\ & & & a_0 \end{pmatrix}$$

. lower triangular Toeplitz matrix

$$\Delta(a) = \begin{pmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix}$$

. S-matrix

$$\begin{pmatrix} 0 & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & 0 \end{pmatrix}$$

. E-matrix

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}$$

, with E: $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_0 \end{pmatrix}$.

. E^--matrix

$$\begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

, with E^-: $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} a_{n-1} \\ a_0 \\ a_1 \\ \vdots \\ a_{n-2} \end{pmatrix}$.

. W-matrix (or IDFT-matrix)

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)^2} \end{pmatrix}$$

, with $w = e^{2\pi i/n}$.

. D(w)-matrix

$$\begin{pmatrix} 1 & & 0 \\ & w & \\ & & \ddots \\ 0 & & w^{n-1} \end{pmatrix}$$

$D = D(e^{2\pi i/n})$

$D' = D(e^{\pi i/n})$.

. $\Lambda(a)$-matrix: a diagonal matrix with diagonal elements

$$\Lambda(a)_{ii} = (\bar{W}a)_i$$

. $W\bar{W} = nI$

$WS = \overline{DW}$

$SW = \overline{WD}$

$DWS = \bar{W}$

$WE^{-} = DW$

$EW = WD$

$WSEq = \overline{Wq}$, q a vector

$WE = \bar{D}W$

$E^{-}W = W\bar{D}$

$\Delta(a) = S\nabla(SEa)S$

$$a' = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \bar{D}'a.$$

. The number of operations of a W-matrix-times-vector is $O(n*\Sigma p_i)$, with $n = \pi(p_i)$ if we apply FFT-like algorithms.

5.2 Matrix-times-vector products

The representation of a circulant in terms of W, and Toeplitz matrices in terms of circulants is given. Indicated is how the DFT can be used with respect to the matrix-times-vector products. Applications to the multiplication of polynomials are treated, where the treatment of the multiplication of two polynomials in the Chebyshev sum representation is possibly new. No implementations are proposed, because the treatment of the solution of linear systems with Toeplitz matrices is not yet clarified. (For the matrix-times-vector products a set of dyadic operators is a realistic possibility).

LEMMA 1. (Eigensystem of circulant)

$$C(c) = n^{-1} W \Lambda(c) \bar{W}.$$

PROOF. Multiplication of the eigensystem equation

$$Cv = \lambda v$$

by $\bar{W}$, yields for the j-th component

$$(\bar{W}Cv)_j = (\bar{W}c)_j \, (\bar{W}v)_j = \lambda (\bar{W}v)_j.$$

The solution of the transformed equation is

$$\lambda = (\bar{W}c)_j$$
$$\bar{W}v = e_j \qquad \text{(the j-th unit vector)}.$$

The equations for the eigensystem

$$CW = W\Lambda$$

give the factorization

$$C = n^{-1} W \Lambda \bar{W}.$$

THEOREM 1. (Circulant times vector)

The product can be represented by

$$C(c)b = n^{-1}(W(\Lambda(c)(\bar{W}b))).$$

PROOF. Apply the decomposition given in lemma 1.

Remarks.

- A product of a rectangular circulant times a vector can be obtained by the above formula by padding the shorter of b and c with zeros.
- A circulant times vector is also called a circular convolution and is often denoted by

$$\sum_{i=0}^{n-1} c_{j-i}b_i, \quad j = 0,1,\ldots \text{ and } c_k = c_{n+k}, \qquad \text{all integer } k.$$

LEMMA 2. (Representation of an upper Triangular Toeplitz matrix as a sum of a circulant and a diagonal similarity transformation of a circulant.)

$$\nabla(a) = \tfrac{1}{2}\{C(a) + D'C(a')\bar{D}'\}.$$

PROOF.

$$\nabla(a) = \tfrac{1}{2}\left\{C(a) + \begin{pmatrix} a_0 & \cdots & a_1 \\ -a_1 & a_0 & \cdots & a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ -a_{n-1} & \cdots & -a_1 & a_0 \end{pmatrix}\right\}$$

with the decomposition of the second term

$$\begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ -a_1 & a_0 & \cdots & a_2 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & & -a_1 & a_0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & 0 \\ & w & \\ & & \ddots \\ 0 & & w_{n-1} \end{pmatrix} \begin{pmatrix} a_0 & wa_{n-1} & \cdots & w^{n-1}a_1 \\ w^{n-1}a_1 & a_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ wa_{n-1} & \cdots & & a_0 \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \bar{w} & \\ & & \ddots \\ 0 & & \bar{w}^{(n-1)} \end{pmatrix}$$

with $w = e^{\pi i/n}$ and $a' = \begin{pmatrix} a_0 \\ w^{n-1}a_1 \\ \vdots \\ wa_{n-1} \end{pmatrix}$.

The result is obtained if we use the notation of 5.1.

Remark. The proof given above is constructive. Once the result is known, a more direct proof can be given by evaluating the sum of the circulants.

THEOREM 2. (Upper triangular Toeplitz matrix times a vector)

$$\nabla(a)\, b = \tfrac{1}{2}\{C(a) + D'C(a')\bar{D}'\}b$$

$$= \{W\Lambda(a)\bar{W}b + D'W\Lambda(a')\bar{W}\bar{D}'b\}/2n).$$

PROOF. Apply the decomposition given in lemma 2.

Remark. For the calculation we need

$\bar{W}b$

$\bar{\bar{W}}\bar{D}'b$

$\bar{W}a$ (for $\Lambda(a)$)

$\bar{W}a'$ (for $\Lambda(a')$).

COROLLARY 1. (The coefficients of the product of two (balanced) polynomials in power sum representation).

Let

$$P_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k \text{ and } Q_{n-1}(x) = \sum_{k=0}^{n-1} b_k x^k$$

then

$$R_{2(n-1)}(x) = P_{n-1}(x)\, Q_{n-1}(x) = \sum_{j=0}^{2(n-1)} c_j x^j$$

with

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \\ \vdots \\ c_{2n-2} \end{pmatrix} = \begin{pmatrix} a_0 & & & & 0 \\ a_1 & a_0 & & & \\ \vdots & & \ddots & & \\ a_{n-1} & \cdots & a_1 & a_0 & \\ & a_{n-1} & & a_1 & \\ 0 & & & & a_{n-1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

or

$$c_j = \sum_{k=0}^{n-1} b_k a^*_{j-k}, \qquad j = 0,1,\ldots,2(n-1)$$

with $a^*_j = a_j$, $j = 0,1,\ldots,n-1$ and $a^*_j = 0$ for $j < 0$, $j \geq n$.

The calculation can be reduced to the upper triangular Toeplitz matrix times vector products

$$\{c_k\}_{n-1}^{2(n-1)} = \nabla(E^{-}a)\, b$$

$$\{c_k\}_{0}^{n-1} = S\nabla(SEa)\,(Sb).$$

The vectors in these products are related

$$E^{-}a \qquad \text{with } E^{-}Sa \qquad (E^{-}S = SE)$$

and

$$b \qquad \text{with } Sb.$$

For the calculation with $a,b \in \mathbb{R}^n$ some products with W can be written as

$$WE^{-}a = DWa;$$
$$WE^{-}Sa = DWSa = \bar{W}a = \overline{Wa}, \qquad \text{for } a \in \mathbb{R}^n;$$
$$Wb;$$
$$WSb = \overline{DW}b = \overline{DWb}, \qquad \text{for } b \in \mathbb{R}^n.$$

COROLLARY 2. (The coefficients of the product of two (balanced) polynomials in Chebyshev sum representation).

Let

$$A_{n-1}(x) = \sum_{k=0}^{n-1} p_k T_k(x) \quad \text{and} \quad B_{n-1}(x) = \sum_{k=0}^{n-1} q_k T_k(x)$$

then

$$C_{2(n-1)}(x) = \sum_{k=0}^{2(n-1)} r_k T_k(x) = \tfrac{1}{2}\{ \sum_{k=0}^{2(n-1)} c_k T_k(x) + \sum_{k=0}^{n-1}{}' (a_k+b_k) T_k(x) \}$$

with

$$\begin{pmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_{n-1} \\ c_n \\ \cdot \\ \cdot \\ \cdot \\ c_{2(n-1)} \end{pmatrix} = \begin{pmatrix} q_0 & & & & 0 \\ q_1 & q_0 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ q_{n-1} & & q_1 & & q_0 \\ & q_{n-1} & & & q_1 \\ 0 & & \cdot & & \\ & & & \cdot & \\ & & & & q_{n-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_{n-1} \end{pmatrix}$$

or $c_j = \sum_{k=0}^{n-1} q^*_{j-k} p_k, \quad j = 0,1,\ldots,2(n-1),$

$$\begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} q_{n-1} & \cdots & q_0 \\ & q_{n-1} & \cdot \\ & & \cdot \\ 0 & & \cdot \\ & & \\ & & q_{n-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_{n-1} \end{pmatrix}$$

or $a_j = \sum_{k=0}^{n-1} q^*_{j+k} p_k, \quad j = 0,1,\ldots,n-1,$

$$\begin{pmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-1} \end{pmatrix} \begin{pmatrix} q_0 & q_1 & \cdots & q_{n-1} \\ & q_0 & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot \\ 0 & & & q_1 \\ & & & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_{n-1} \end{pmatrix}$$

or $b_j = \sum_{k=0}^{n-1} q^*_{k-j} P_k \quad (= \sum_{k=0}^{n-1} P^*_{j+k} q_k), \qquad j = 0,1,\ldots,n-1;$

the vectors with asterisk denote the vectors without asterisk padded with zeroes.

PROOF.

From

$$\sum_{k=0}^{n-1} p_k T_k(x) \sum_{k=0}^{n-1} q_k T_k(x) = \tfrac{1}{2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} p_k q_\ell \{T_{k+\ell}(x) + T_{|k-\ell|}(x)\}$$

$$= \tfrac{1}{2}\{ \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} p_k q_\ell T_{k+\ell}(x) + \sum_{k=0}^{n-1} \sum_{\ell=k}^{n-1}{}' p_k q_\ell T_{\ell-k}(x) +$$

$$\sum_{\ell=0}^{n-1} \sum_{k=\ell}^{n-1}{}' p_k q_\ell T_{k-\ell}(x)\}.$$

The first term can be represented by substitution of $j = k+\ell$ as

$$\sum_{j=0}^{2(n-1)} (\sum_{k=0}^{n-1} q^*_{j-k} p_k) T_j(x).$$

The second term can be represented by substitution of $j = \ell-k$ as

$$\sum_{j=0}^{n-1}{}' (\sum_{k=0}^{n-1} q^*_{j+k} p_k) T_j(x) \text{ or } \sum_{j=0}^{n-1}{}' (\sum_{k=0}^{n-1} q^*_{j+n-1-k} p_{n-1-k}) T_j(x).$$

The third term can be represented by substitution of $j = k-\ell$ as

$$\sum_{j=0}^{n-1}{}' (\sum_{\ell=0}^{n-1} p^*_{j+\ell} q_\ell) T_j(x) \text{ or } \sum_{j=0}^{n-1} (\sum_{\ell=0}^{n-1} q^*_{\ell-j} p_\ell) T_j(x).$$

Remark.

The calculation can be reduced to

$$\{c_k\}_{n-1}^{2(n-1)} = \nabla(E^- q)\ p$$

$$\{c_k\}_0^{n-1} = S\nabla(SEq)\ Sp$$

$$a = \nabla(E^- q)\ Sp$$

$$b = \nabla(SEq)\ p$$

Again the vectors in these products are related

$$E^{-}q \text{ with } SEq$$

and

$$p \quad \text{with } Sp.$$

For the calculation with $p,q \in \mathbb{R}^n$ some products with W can be written as

$$\begin{aligned} WE^{-}q &= DWq \\ WSEq &= \overline{DWEq} = \overline{Wq} \\ WSq &= \overline{DWq}. \end{aligned}$$

LEMMA 3. (representation of a Toeplitz matrix as a difference of a circulant and a diagonal similarity transformation of a circulant

$$T(t) = C((t+E^{-}St_{-})/2) - D'C((t-E^{-}St_{-})'/2)\bar{D}'.$$

PROOF.

$$T(t) = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{n-1} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & t_{-1} \\ t_{n-1} & \cdots & & t_0 \end{pmatrix} =$$

$$= C(t) + \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-(n-1)} \\ & t_0 & \ddots & \vdots \\ & & \ddots & t_{-1} \\ 0 & & & t_0 \end{pmatrix} - \begin{pmatrix} t_0 & t_{n-1} & \cdots & t_1 \\ & t_0 & \ddots & \vdots \\ & & \ddots & t_{n-1} \\ 0 & & & t_0 \end{pmatrix}$$

$$= C(t) + \nabla(u) - \nabla(t)$$

with

$$u = \begin{pmatrix} t_0 \\ t_{-(n-1)} \\ \cdot \\ \cdot \\ \cdot \\ t_{-1} \end{pmatrix} = E^{-}St_{-}.$$

From the representation of an upper triangular Toeplitz matrix in lemma 2, we have

$$C(t) - \nabla(t) = C(t) - \tfrac{1}{2}\{C(t) + D'C(t')\bar{D}'\}$$
$$= \tfrac{1}{2}\{C(t) - D'C(t')\bar{D}'\}.$$

Therefore,

$$T(t) = \tfrac{1}{2}\{C(t) - D'C(t')\bar{D}'\} +$$
$$\tfrac{1}{2}\{C(E^{-1}St_{-}) + D'C((E^{-}St_{-})')\bar{D}'\}$$
$$= C((t+E^{-}St_{-})/2) - D'C((t-E^{-}St_{-})'/2)\bar{D}'.$$

Remark. A band Toeplitz matrix with u upper and ℓ lower codiagonals (ℓ+u<n) can be represented by the sum of a circulant of order n and a lower triangular Toeplitz matrix of order u and an upper triangular Toeplitz matrix of order ℓ.

THEOREM 3. (Toeplitz matrix times vector)

$$T(t)b = \{W\Lambda(t+E^{-}St_{-})\bar{W}b - D'W\Lambda((t-E^{-}St_{-1})')\bar{W}\bar{D}'b\}/(2n).$$

PROOF.

Apply the factorization of a circulant as given in lemma 1 to the circulants in the representation of a Toeplitz matrix as given in lemma 3.

Remarks.

. For the calculation we need

$$\bar{W}b = \overline{Wb} \qquad \text{for } b \in \mathbb{R}^n$$

$$\bar{W}\bar{D}'b = \overline{WD'b}, \quad \text{for } b \in \mathbb{R}^n$$

$$\bar{W}(t+\bar{E}\bar{S}t_{\_}) = \overline{W(t+\bar{E}\bar{S}t_{\_})}, \quad \text{for } t, t_{\_} \in \mathbb{R}^n$$

$$\bar{W}((t-\bar{E}\bar{S}t_{\_})') = \overline{W((t-\bar{E}\bar{S}t_{\_})')}.$$

. A Hankel matrix times vector can be reduced to the above case because

$$H = ST.$$

. High speed convolution (correlation) is merely a Toeplitz(Hankel)-matrix-times-vector product via the above factorizations.

5.3 Solution of linear systems

The theorems in this paragraph handle the possibility to obtain the solution fast; exceptional cases and algorithmic details are not yet available.

THEOREM 4. (Solution of a linear system with a circulant as matrix)

$$C(c)x = b \iff x = n^{-1}W(\Lambda^{-1}(c)(\bar{W}b)).$$

provided $\Lambda(c)$ is not singular.

PROOF. From theorem 1 we have

$$C(c)x = n^{-1}W\Lambda(c)\bar{W}x = b$$

and therefore

$$x = n^{-1}W\Lambda^{-1}(c)\bar{W}b.$$

Remarks

. Berg (1975) proposed to use the easy solution of a linear system with a circulant matrix C, for a general linear system with matrix A, by splitting

$$A = C-D.$$

$$Ax = b \iff x_{k+1} = C^{-1}b + C^{-1}Dx_k, \qquad k = 0,1,\ldots .$$

with

$$\|C^{-1}D\| < 1.$$

The iteration can be modified for singular C.

. According to regular splittings a wealth of literature has emerged, see e.g. Berman & Plemmons (1974) and Neumann (1976).

THEOREM 5. (Solution of a linear system with an upper triangular Toeplitz matrix).

The solution of a linear system

$$\triangledown(a)x = b \iff \begin{pmatrix} \triangledown & T \\ & \triangledown \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ \cdots \\ b_2 \end{pmatrix}$$

can recursively be reduced to the solution of smaller systems of the same structure and Toeplitz-matrix-times-vector products.

PROOF. Partitioning of $\triangledown$ yields the result.

Remarks.

. The normal back substitution

$$x_{n-1} = b_{n-1}/a_0$$
$$x_{n-k} = b_{n-k} - \sum_{j=1}^{k} a_j x_{n-k+j}, \qquad k = 2,3,\ldots,n-1$$

takes

$$0(n^2) \text{ operations.}$$

. The problem of theorem 5 occurs by polynomial division as can be seen from corollary 1 (and 2 with a rank one update)
determine b, say, from

$$\nabla(\bar{E}a)\ b = \{c_k\}_{n-1}^{2(n-1)} .$$

. The problem of theorem 5 occurs by the determination of the inverse of an upper triangular Toeplitz matrix, because the inverse is again an upper triangular Toeplitz matrix (Kung (1973; theorem 2.3), Derr (1971))

$$\nabla(a)x = e^{(n)} \iff \nabla(a)\ \nabla(\bar{E}x) = I.$$

. Theorem 5 gives the general idea of how to solve this system of equations; the details of the implementation are not yet worked out.

THEOREM 6. (Solution of a linear system with a band Toeplitz matrix with ℓ lower and u upper codiagonals, Berg (1975))

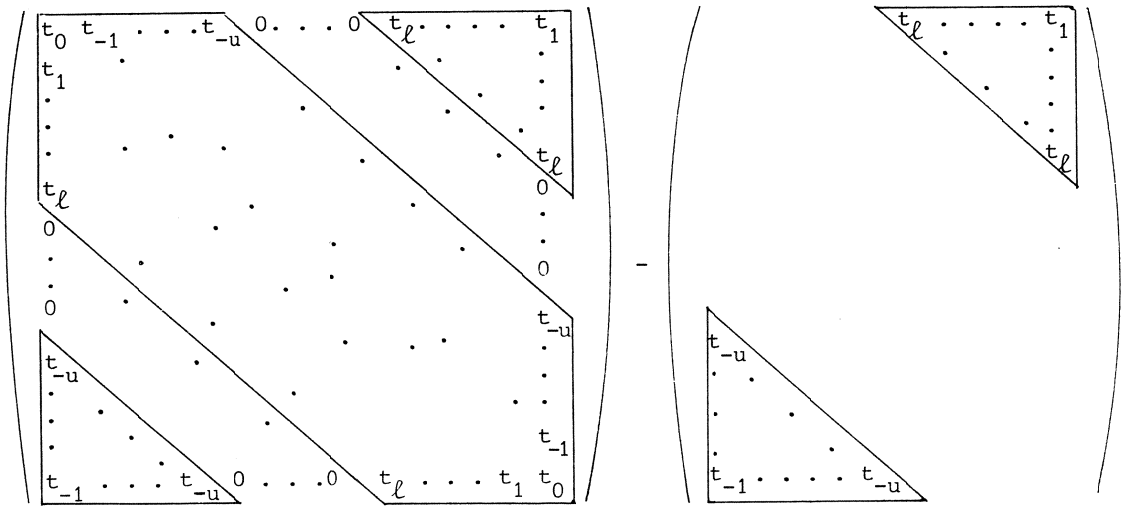
The solution of a band Toeplitz system of equations with ℓ lower codiagonals and u upper codiagonals (ℓ+u<n) can be reduced to the solution of a system of ℓ+u equations, provided the circulant is regular.

PROOF.

The band Toeplitz matrix

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-u} & 0 & \cdots & 0 \\ t_1 & t_0 & \ddots & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ t_\ell & & \ddots & \ddots & \ddots & & t_{-u} \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & & \ddots & t_0 & t_{-1} \\ 0 & \cdots & \cdots & t_\ell & \cdots & t_1 & t_0 \end{pmatrix}$$

can be split into



The linear equations

$$Tx = b$$

can be split into

$$(C - (\ \ \ ))x = b\ ,$$

and for regular C - the above given circulant -

$$x - C^{-1}(\ \ \ )x = C^{-1}b.$$

If we call

$$\hat{x} = (\ \ \ )x \quad \text{(a function of } x_1,\ldots,x_u,x_{n-\ell+1},\ldots x_n)$$

then we arrive at the ℓ+u system of equations

$$x_k - (C^{-1}\hat{x})_k = (C^{-1}b)_k, \qquad k = 1,\ldots,u, \text{ and } n-\ell+1,\ldots,n.$$

The resulting components are obtained from

$$x_k = (C^{-1}(b+\hat{x}))_k, \qquad k = u+1,\ldots,n-\ell.$$

Remarks.

. The above technique is of considerable importance for block band Toeplitz matrices, as arise from discretization of partial differential equations, e.g. the Poisson equation.

. Henrici (1979) considers a more dimensional circular convolution, and its properties under the DFT analogous to the one-dimensional convolution, as the central point. Two-dimensional recurrence relations with constant coefficients and block band Toeplitz matrices are particular cases of a two-dimensional convolution, and by a proper extension circular convolutions (this last aspect simplifies the problem of inversion and is "given" by the problem).
On the other hand it is interesting to consider a block Toeplitz matrix and to think of solutions of a Toeplitz system of equations where the elements are again Toeplitz matrices, so the multiplication and 'division' are performed on operands of type Toeplitz matrix and vector.

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V.6 Consideration of the Winograd technique for the improvement of the DFT.

The calculation of

$$A_k = \sum_{\ell=0}^{n-1} a_\ell \exp(-2\pi i k\ell/n), \qquad k = 0,1,\ldots,n-1$$

with n prime, can be reduced to

$$A_k - a_0 = \sum_{\ell=1}^{n-1} a_\ell \exp(-2\pi i k\ell/n), \quad k = 1,2,\ldots,n-1.$$

Because n is prime, the numbers 1,2,...,n-1 form a cyclic group with g as primitive root, say (Abramowitz & Stegun, p. 827). Therefore a permutation in the summation, with notation ((x)) = x modulo n,

$$\ell \to ((g^{\ell})),$$
$$k \to ((g^{k}))$$

yields

$$A_{((g^k))} - a_0 = \sum_{\ell=0}^{n-2} a_{((g^\ell))} \exp(-2\pi i g^{\ell+k}/n)$$

(Note that $((g^{n-1})) = g^0$.)

This summation is a circular correlation of size n-1; this circulant-times-vector can be calculated via theorem 1 (with Λ precomputed).

Although we consider this the principle of the Winograd technique it requires some more detailed study, whether this is the Winograd technique or not; the necessity of modification of C06FFT depends on the relation between increase of the bookkeeping and lower intrinsic computational complexity.

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